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RAY'S NEW HIGHER ALGEBRA.

ELEMENTS

OF

ALGEBRA,

FOR

COLLEGES, SCHOOLS, AND PRIVATE STUDENTS.

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PREFACE.

ALGEBRA is justly regarded one of the most interesting and useful branches of education, and an acquaintance with it is now sought by all who advance beyond the more common elements. To those who would know Mathematics, a knowledge not merely of its elementary principles, but also of its higher parts, is essential; while no one can lay claim to that discipline of mind which education confers, who is not familiar with the logic of Algebra.

It is both a demonstrative and a practical science—a system of truths and reasoning, from which is derived a collection of Rules that may be used in the solution of an endless variety of problems, not only interesting to the student, but many of which are of the highest possible utility in the arts of life.

The object of the present treatise is to present an outline of this science in a brief, clear, and practical form. The aim throughout has been to demonstrate every principle, and to furnish the student the means of understanding clearly the *rationale* of every process he is required to perform. No effort has been made to simplify subjects by omitting that which is difficult, but rather to present them in such a light as to render their acquisition within the reach of all who will take the pains to study.

To fix the principles in the mind of the student, and to show their bearing and utility, great attention has been paid to the preparation of practical exercises. These are intended rather to illustrate the principles of the science, than as difficult problems to torture the ingenuity of the learner, or amuse the already skillful Algebraist.

An effort has been made throughout the work to observe a natural and strictly logical connection between the different parts, so that the learner may not be required to rely on a prin-

ciple, or employ a process, with the *rationale* of which he is not already acquainted. The reference by Articles will always enable him to trace any subject back to its first principles.

The limits of a preface will not permit a statement of the peculiarities of the work, nor is it necessary, as those who are interested to know will examine it for themselves. It is, however, proper to remark, that Quadratic Equations have received more than usual attention. The same may be said of Radicals, of the Binomial Theorem, and of Logarithms, all of which are so useful in other branches of Mathematics.

On some subjects it was necessary to be brief, to bring the work within suitable limits. For example, what is here given of the Theory of Equations, is to be regarded merely as an outline of the more practical and interesting parts of the subject, which alone is sufficient for a distinct treatise, as may be seen by reference to the works of Young or Hymers in English, or of DeFourcy or Reynaud in French.

Some topics and exercises, deemed both useful and interesting, will be found here, not hitherto presented to the notice of students. But these, as well as the general manner of treating the subject, are submitted, with deference, to the intelligent educational public, to whom the author is already greatly indebted for the favor with which his previous works have been received.

WOODWARD COLLEGE, *May*, 1852.

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HIGHER ALGEBRA.

I. DEFINITIONS.

Article 1. Mathematics is the science of the exact relations of Quantity as to Magnitude or Form.

2. Quantity, as the subject of mathematical investigations, is any thing capable of being measured, or about which the question How much? may be asked. It may be, 1. Geometric, involving Form; 2. Number.

3. Number is quantity considered as composed of equal parts of the same kind, each called the *unit;* and the *magnitude* of the quantity is indicated by its ratio to the unit.

4. Numbers are represented by conventional symbols. When the symbols used are *general,* as distinguished from the arithmetical symbols, viz., the Arabic numerals, the process of investigation is called *Algebraic.* Hence, we have the following definitions:

5. Algebra is the method of investigating the relations of numbers by means of *general* symbols.

REMARK.—It should be remembered that the word "*quantity,*" whenever used in algebra, is synonymous with "*number.*"

6. The algebraic symbols are of two kinds: 1. Symbols of numbers; 2. Symbols of relation.

Numbers are usually represented by letters; as, a, b, x, y: sometimes, of course, when known, by the Arabic numerals.

7. The symbols of relation, usually called **Signs,** are the representatives of certain phrases, and are used to express operations with precision and brevity. The principal algebraic signs are: $= \; + \; - \; \times \; \div \; \sqrt{}$.

8. The **Sign of Equality,** $=$, is read *equal to.* It denotes that the quantities between which it is placed are equal. Thus, $x=5$, denotes that the quantity represented by x equals 5.

9. The **Sign of Addition,** $+$, is read *plus.* It denotes that the quantity to which it is prefixed is to be added. Thus, $a+b$ denotes that b is to be added to a.

10. The **Sign of Subtraction,** $-$, is read *minus.* It denotes that the quantity to which it is prefixed is to be subtracted. Thus, $a-b$ denotes that b is to be subtracted from a.

11. The signs $+$ and $-$ are called *the* signs. The former is called the *positive*, the latter the *negative* sign; they are said to be *contrary*, or *opposite.*

12. Every quantity is supposed to have either the positive or negative sign. When a quantity has no sign prefixed to it, $+$ is understood. Thus, $a=+a$.

Quantities having the positive sign are called *positive;* those having the negative sign, *negative.*

13. Quantities having the same sign are said to have *like* signs; those having different signs, *unlike* signs.

Thus, $+a$ and $+b$, or $-a$ and $-b$, have like signs; while $+c$ and $-d$ have unlike signs.

14. The **Sign of Multiplication,** $\times$, is read *into*, or *multiplied by.* It denotes that the quantities between which it is placed are to be multiplied together.

The product of two or more letters is also expressed by a dot or period, or by writing the letters in close succession. The last method is generally to be preferred.

Thus, the continued product of the numbers designated by a, b, and c, is denoted by $a\times b\times c$, or $a.b.c$, or abc.

15. Factors are quantities that are to be multiplied together. Thus, in the product ab, there are two factors, a and b; in the product $3\times5\times7$, there are three factors, 3, 5, and 7.

16. The **Sign of Division,** $\div$, is read *divided by.* It denotes that the quantity preceding it is to be divided by that following it.

Division is also expressed by placing the dividend as the numerator, and the divisor as the denominator of a fraction.

Thus, $a\div b$, or $\frac{a}{b}$, signifies that a is to be divided by b.

17. The **Sign of Inequality,** $>$, denotes that one of the two quantities between which it is placed is greater than the other. The *opening* of the sign is toward the greater quantity.

Thus, $a>b$, denotes that a is greater than b. It is read *a greater than b.* Also, $c<d$, denotes that c is less than d, and is read, *c less than d.*

18. A **Coefficient** is a number or letter prefixed to a quantity, to show how many times it is taken.

Thus, if a is to be taken 4 times, instead of writing $a+a+a+a$, write $4a$; also, $az+az+az=3az$.

The coëfficient is called *numeral* or *literal*, according as it is a number or a letter. Thus, in the quantities $5x$ and mx, 5 is a numeral and m a literal coëfficient.

In $3az$, 3 may be considered as the coëfficient of az, or $3a$ as the coëfficient of z.

When no numeral coëfficient is expressed, **1** is understood. Thus, a is the same as $1a$, and ax the same as $1ax$.

19. A **Power** of a quantity is the product arising from multiplying the quantity by itself one or more times.

When the quantity is taken *twice* as a factor, the product is called the *second* power; when *three* times, the *third* power; and so on.

Thus, $2\times2=4$, is the *second* power of 2.
$2\times2\times2=8$, is the *third* power of 2.
$2\times2\times2\times2=16$, is the *fourth* power of 2.

Also, $a\times a=aa$, is the *second* power of a.
$a\times a\times a=aaa$, is the *third* power of a; and so on.

The second power is often termed the *square*, and the third power, the *cube*.

An **Exponent** is a small figure or letter placed on the right, and a little above a quantity, to express its power.

Thus, $aa=a^2$, $aaa=a^3$, etc. a^m indicates that a is taken as a factor as many times as there are units in m.

When no exponent is expressed, 1 is understood. Thus, a is the same as a^1, each signifying the *first* power of a.

20. A **Root** of a quantity is a factor, which, multiplied by itself a certain number of times, will produce the given quantity.

The root is called the *square* or *second* root, the *cube* or *third* root, the *fourth* root, etc., according to the number of times it must be taken as a factor to produce the given quantity.

Thus, 2 is the second or square root of 4, since $2\times2=4$; a is the fourth root of a^4, since $a\times a\times a\times a=a^4$.

21. The **Radical Sign,** $\sqrt{}$ or $\sqrt{}$, when prefixed to a quantity, denotes that its root is to be extracted.

An **Index** is a figure or letter placed over a radical sign to denote what root is to be taken. Thus,

$\sqrt[2]{}9$, or $\sqrt[2]{9}$, denotes the square root of 9, which is 3.
$\sqrt[3]{}8$, or $\sqrt[3]{8}$, denotes the cube root of 8, which is 2.
$\sqrt[4]{}a$, or $\sqrt[4]{a}$, denotes the fourth root of a

When the radical sign has no index over it, 2 is understood; thus, $\sqrt{}a$ and $\sqrt[2]{}a$ signify the same thing.

22. An **Algebraic Quantity,** or an *Algebraic Expres-*

sion, is any quantity written in algebraic language, that is, by means of symbols. Thus,

$5a$, is the algebraic expression of 5 times the number a;
$3b+4c$, is the algebraic expression for 3 times the number b increased by 4 times the number c;
$3a^2-7ab$, for 3 times the square of a, diminished by 7 times the product of the number a by the number b.

23. A **Monomial** is a quantity not united to any other by the sign of addition or subtraction; as, $4a$, a^2bc, $-4xy$, etc.

A monomial is often called a *simple quantity*, or *term*.

24. A **Polynomial,** or *Compound Quantity*, is an algebraic expression composed of two or more terms; as, $a+b$, $c-x+y$, etc.

25. A **Binomial** is a quantity having two terms; as, $a+b$, x^2+y, etc.

A **Residual Quantity** is a binomial, the second term of which is negative; as, $a-b$.

26. A **Trinomial** is a quantity consisting of three terms; as, $a+b-c$.

27. The **Numerical Value** of an algebraic expression is the number obtained by giving a particular value to each letter, and then performing the operations indicated.

Thus, in the algebraic expression $4a-3c$, if $a=5$ and $c=6$, the numerical value is $4\times5-3\times6=20-18=2$.

28. The value of a polynomial is not affected by changing the order of the terms, provided each term retains its sign.

Thus, $b^2-2ab+c$ is evidently the same as $b^2+c-2ab$.

29. The **Degree** of any term is equal to the number of *literal* factors which it contains.

Thus, $5a$ is of the *first* degree; it contains *one* literal factor.
ax is of the *second* degree; it contains *two* literal factors.
$3a^3b^2c=3aaabbc$, is of the *sixth* degree.

30. A polynomial is said to be *homogeneous* when each of its terms is of the same degree. Thus,

$a+b-3c$ is homogeneous; each term being of the first degree.
x^3-7xy^2 is homogeneous; each term being of the third degree.
x^2-3xy^2 is not homogeneous.

31. An algebraic quantity is said to be *arranged* according to the dimensions of any letter it contains, when the exponents of that letter occur in the order of their magnitudes, either *increasing* or *decreasing*.

Thus, $ax^2+a^2x-a^3x^3$, is arranged according to the ascending powers of a; and $bx^3-b^3x^2+b^2x$, is arranged according to the descending powers of x.

32. A **Parenthesis,** ( ), is used to show that all the terms of a polynomial which it incloses are to be considered together, as a single term. Thus,

$10-(a-b)$ means that $a-b$ is to be subtracted from 10.
$5(a+b-c)$ means that $a+b-c$ is to be multiplied by 5.
$5a+(b-c)$ means that $b-c$ is to be added to $5a$.

When the parenthesis is used, the sign of multiplication is generally omitted. Thus, $(a-b)\times(a+b)$, is written $(a-b)(a+b)$.

A **Vinculum,** ———, is sometimes used instead of a parenthesis. Thus, $\overline{a+b}\times5$ means the same as $5(a+b)$. Sometimes the vinculum is placed vertically; it is then called a *bar*.

Thus, $\left.\begin{array}{r} a \\ -b \\ +c \end{array}\right| x^2$, is the same as $(a-b+c)x^2$.

33. Similar, or **Like Quantities,** are such as contain the same letter or letters with the same exponents.

Thus, $3ab$ and $-2ab$, $3a^2b$ and $5a^2b$, $3a^2b$ and $-5a^2b$, are similar.

Unlike Quantities are such as contain different letters or *different powers* of the *same* letter.

Thus, $5a$ and $3b$, $3ab^2$ and $3a^2b$, are unlike or dissimilar.

REMARK.—An exception must be made in those cases where letters are taken to represent coëfficients. Thus, ax^2 and bx^2 are like quantities, when a and b are taken as coëfficients of x^2.

34. The **Reciprocal** of a quantity is unity divided by that quantity. Thus,

The reciprocal of a is $\frac{1}{a}$; of 3, is $\frac{1}{3}$; of $\frac{2}{3}$, is $1 \div \frac{2}{3} = \frac{3}{2}$; Hence,

The reciprocal of a fraction is the fraction inverted.

35. The same letter, accented, is often used to denote quantities which occupy similar positions in different equations or investigations.

Thus, a, a', a'', a''', read, a, a prime, a second, a third, and so on.

36. The following signs are also used, for the sake of brevity:

∞, a quantity indefinitely great, or infinity.

$\therefore$, signifies *therefore*, or *consequently*.

$\because$, signifies *since*, or *because*.

$\sim$, is used to represent the difference between two quantities, as $c \sim d$, when it is not known which is the greater.

EXERCISES.

First, copy each example on the slate or blackboard; and then *read* it, that is, express it in common language.

Second, find the numerical value in each, supposing $a=2$, $b=3$, $c=4$, $x=5$, $y=6$.

1. $7b+x-y$. Ans. 20.
2. a^2by-3x^2. Ans. -3.
3. $c+a\times c-a$. Ans. 10.
4. $(c+a)(c-a.)$ Ans. 12.
5. $\frac{a^2+b+c-y}{2}$. Ans. $2\frac{1}{2}$.
6. $\frac{cx-ay}{x-b}$. Ans. 4.
7. $\frac{bc}{y}+\frac{ay}{c}$. Ans. 5.
8. $\frac{ab(c-a)}{y-c}-\sqrt{aby}$. Ans. 0.

9. Find the difference between abx, and $a+b+x$, when $a=4$, $b=\frac{1}{2}$, $x=3$; and when $a=5$, $b=7$, $x=12$. Ans. $1\frac{1}{2}$ and 396.

10. Required the values of $a^2+2ab+b^2$, and $a^2-2ab+b^2$, when $a=7$ and $b=4$. Ans. 121 and 9.

11. What is the value of $n(n-1)(n-2)(n-3)$, when $n=4$, and when $n=10$? Ans. 24 and 5040.

12. Find the difference between $6abc-2ab$, and $6abc\div 2ab$, when a, b, c, are 3, 5, and 6 respectively. Ans. 492.

13. Find the value of $\frac{a^2-b^2}{a^3+b^3}$, when $a=5$ and $b=3$.

Ans. $\frac{2}{19}$.

Verify the following, by giving to each letter any value whatever:

14. $a(m+n)(m-n)=am^2-an^2$.

15. $\frac{x^3-y^3}{x-y}=x^2+xy+y^2$.

TO BE EXPRESSED IN ALGEBRAIC SYMBOLS.

1. Five times a, plus the second power of b.
2. x, plus y divided by $3z$.
3. x plus y, divided by $3z$.
4. 3 into x minus n times y, divided by m minus n.
5. a third power minus x third power, divided by a second power minus x second power.
6. The square root of m minus the square root of n.
7. The square root of m minus n.

ANSWERS.

1. $5a+b^2$.	3. $\frac{x+y}{3z}$.	5. $\frac{a^3-x^3}{a^2-x^2}$.
2. $x+\frac{y}{3z}$.	4 $\frac{3x-ny}{m-n}$.	6. $\sqrt{m}-\sqrt{n}$.
		7. $\sqrt{(m-n)}$.

ADDITION.

37. Addition, in Algebra, is the process of finding the simplest expression for the sum of two or more algebraic quantities.

There are three cases of algebraic addition:

1st. When the quantities are similar, and have like signs.

2d. When the quantities are similar, but the signs unlike.

3d. When the quantities are dissimilar, or part similar and part dissimilar.

38. First Case.—Let it be required to find the sum of $3x^2y$, $5x^2y$, and $7x^2y$.

Here, x^2y is taken, in the first term, 3 times; in the second, 5 times; and in the third, 7 times; hence, in all, it is taken 15 times. Since adding the quantities can not change their character, and since each term is positive, their sum is positive.

OPERATION.

$$\begin{array}{r} +\ 3x^2y \\ +\ 5x^2y \\ +\ 7x^2y \\ \hline +15x^2y \end{array}$$

Find the sum of $-3x^2y$, $-5x^2y$, and $-7x^2y$.

Here, x^2y is taken, in the first term, -3 times; in the second, -5 times; and in the third, -7 times; hence, in all, it is taken -15 times. Therefore, to add similar quantities having the same sign,

OPERATION.

$$\begin{array}{r} -\ 3x^2y \\ -\ 5x^2y \\ -\ 7x^2y \\ \hline -15x^2y \end{array}$$

Rule.—*Add the coëfficients, and prefix the sum, with the common sign, to the literal part.*

39. Second Case.—Let it be required to find the sum of $+9a$, $-5a$, $+4a$, and $-2a$.

Here, $+9a+4a$ is $+13a$; and $-5a-2a$ is $-7a$. Now, since the sum of two equal quantities, of which one is positive and the other negative, is evidently 0, $-7a$ will cancel $+7a$ in the quantity $+13a$, and leave $+6a$ for the *aggregate*, or result of the four quantities.

OPERATION.

$$\begin{array}{r} +9a \\ -5a \\ +4a \\ -2a \\ \hline +6a \end{array}$$

In like manner, to obtain the sum of $-9a$, $+5a$, $-4a$, and $+2a$, we find the sum of $-9a$ and $-4a$ is $-13a$, and the sum of $+5a$ and $+2a$ is $+7a$. Now, $+7a$ will cancel $-7a$ in the quantity $-13a$; which leaves $-6a$ for the aggregate. Therefore,

OPERATION.

$$\begin{array}{r} -9a \\ +5a \\ -4a \\ +2a \\ \hline -6a \end{array}$$

TO ADD SIMILAR QUANTITIES HAVING DIFFERENT SIGNS,

Rule.—1. *Add the positive and negative coëfficients separately.*

2. *Subtract the less sum from the greater, and give to the difference the sign of the greater.*

3. *Prefix this difference to the literal part.*

40. Third Case.—Let it be required to find the sum of $5a^2-8b+c$, $+b-a^2$, and $5b+3a^2$.

In writing the quantities, we place, for *convenience*, those which are similar under each other.

The sum in the first column is $+7a^2$, and in the second, $-2b$; there being no term similar to c, it is annexed, with its proper sign.

OPERATION.

$$\begin{array}{r} 5a^2-8b+c \\ -\ a^2+\ b \\ 3a^2+5b \\ \hline 7a^2-2b+c \end{array}$$

41. From the preceding, we derive the following

GENERAL RULE FOR ADDITION OF ALGEBRAIC QUANTITIES.

1. *Write the quantities to be added, placing those that are similar under each other.*

2. *Add the similar quantities by the rules already given.*

3. *Annex the other quantities with their proper signs.*

REMARK.—In algebraic Addition, Subtraction, and Multiplication, it is best to begin the operation at the left hand.

1. Find the sum of $4ax+3by$, $5ax+8by$, $8ax+6by$, and $20ax+by$. Ans. $37ax+18by$.

2. Of $10cz-2ax^2$, $15cz-3ax^2$, $24cz-9ax^2$, and $3cz-8ax^2$. Ans. $52cz-22ax^2$.

3. Of $3x^2y^2-10y^4$, $-x^2y^2+5y^4$, $8x^2y^2-6y^4$, and $4x^2y^2+2y^4$. Ans. $14x^2y^2-9y^4$.

4. Of $a+b+c+d$, $a+b+c-d$, $a+b-c+d$, $a-b+c+d$, and $-a+b+c+d$. Ans. $3a+3b+3c+3d$.

5. Of $3(x^2-y^2)$, $8(x^2-y^2)$, and $-5(x^2-y^2)$. Ans. $6(x^2-y^2)$.

6. Of $10a^2b-12a^3bc-15b^2c^4+10$, $-4a^2b+8a^3bc-10b^2c^4-4$, $-3a^2b-3a^3bc+20b^2c^4-3$, and $2a^2b+12a^3bc+5b^2c^4+2$. Ans. $5a^2b+5a^3bc+5$.

7. Of $a^m-b^n+3x^p$, $2a^m-3b^n-x^p$, and $a^m+4b^n-x^q$. Ans. $4a^m+2x^p-x^q$.

SUBTRACTION.

42. Subtraction, in Algebra, is the process of finding the difference between two algebraic quantities.

The quantity to be subtracted is called the *subtrahend;* that from which the subtraction is to be made, the *minuend;* the quantity left, the *difference* or *remainder.*

The explanation of the principles on which the operations depend, may be divided into two cases.

1st. Where all the terms are positive.

2d. Where the terms are either partly or wholly negative.

43. First Case.—Let it be required to subtraet $4a$ from $7a$.

It is evident that 7 times any quantity, less 4 times that quantity, is equal to 3 times the quantity; therefore, $7a$ *less* $4a$ is equal to $3a$.

OPERATION.

$7a$	Minuend.
$4a$	Subtrahend.
$3a$	Remainder.

If it be required to subtract b from a, we can only *indicate* the operation, by placing the sign *minus* before the quantity to be subtracted.

OPERATION.

a	Minuend.
b	Subtrahend.
$a—b$	Remainder.

44. Second Case.—Let it be required to subtract $b—c$ from a.

If we subtract b from a, the result, $a—b$, is obviously too little, for the quantity b ought to be diminished by c before it is taken from a. We have, in fact, subtracted a quantity too great by c, and to obtain a true result, the difference, $a—b$, must be increased by c; this gives, for the true remainder, $a—b+c$.

OPERATION.

a	Minuend.
$b—c$	Subtrahend.
$a—b+c$	Remainder.

To illustrate the above example by figures, let $a=9$, $b=5$, and $c=3$; and let it be required to subtract 5—3 from 9.

The operation and illustration may be compared, thus:

From a	From 9	$=9$
Take $b—c$	Take 5—3 . . .	$=2$
Rem. $a—b+c$	Rem. 9—5+3 . . .	$=7$

In the examples already explained, the same result would have been obtained by changing the signs of the quantity to be subtracted, and then adding it.

45. From the preceding, we derive the following

RULE FOR SUBTRACTION OF ALGEBRAIC QUANTITIES.

1. *Write the quantities, placing similar terms under each other.*

2. *Conceive the signs of all the terms of the subtrahend to be changed, from + to —, or from — to +, and then proceed by the rule for algebraic addition.*

REMARK.—1. Beginners should solve a few examples by *actually* changing the signs of the subtrahend.

2. *Proof.—Add the remainder and the subtrahend, as in arithmetic.*

(1)

$$\begin{array}{ll} \text{From} & 8a^2b-3cx-\ z^2 \\ \text{Take} & 3a^2b+4cx-3z^2 \\ \hline \text{Rem.} & 5a^2b-7cx+2z^2 \end{array}$$

The same, with the signs of the subtrahend changed.

(1)

$$\begin{array}{r} 8a^2b-3cx-\ z^2 \\ -3a^2b-4cx+3z^2 \\ \hline \text{Rem. } 5a^2b-7cx+2z^2 \end{array}$$

(2)

$$\begin{array}{ll} \text{From} & 5a^3-3mz+5y^4 \\ \text{Take} & -2a^3+3mz+6y^4 \\ \hline \text{Rem.} & 7a^3-6mz-\ y^4 \end{array}$$

(3)

$$\begin{array}{ll} \text{From} & ax^2-3cy^2-z^2 \\ \text{Take} & bx^2-3cy^2+y^3 \\ \hline \text{Rem.} & (a-b)x^2-\ y^3-z^2 \end{array}$$

4. From $4a-2b+3c$ take $3a+4b-c$.
Ans. $a-6b+4c$.

5. From $9x^2-4y+9$ take $7x^2+5y-14$.
Ans. $2x^2-9y+23$.

6. From $23xy^2-7y+11x^2$ take $11xy^2-5y-9x^2$.
Ans. $12xy^2-2y+20x^2$.

7. From $12x+18$ take $12x-18+y$. Ans. $36-y$.

8. From x^2-y^3 take $-4-y^3+4x^2$. Ans. $4-3x^2$.

9. From $4ax^3+bx+c$ take $3x^3-2x+5$.
Ans. $(4a-3)x^3+(2+b)x+c-5$.

10. From $-17x^3+9ax^2-7a^2x+15a^3$ take $-19x^3+9ax^2-9a^2x+17a^3$. Ans. $2x^3+2a^2x-2a^3$.

11. From x^3+3x^2+3x+1 take x^3-3x^2+3x-1.
Ans. $6x^2+2$.

12. From $9a^mx^2-13+20ab^3x-4b^mcx^2$ take $3b^mcx^2+9a^mx^2-6+3ab^3x$. Ans. $17ab^3x-7b^mcx^2-7$.

13. From $4a^m+2x^p-x^q$ take $a^m-b^n+3x^p$ and $2a^m-3b^n-x^p$. Ans. $a^m+4b^n-x^q$.

The Bracket, or Vinculum.—As the Bracket, or Vinculum, is frequently employed in relation to Addition and Subtraction, it is important that the rules, which govern its use, should be well understood.

46. 1st. *Where the sign plus precedes a parenthesis, or vinculum, it may be omitted without affecting the expression.*

This is self-evident, as is also the converse, viz.:

Any number of terms may be inclosed within a parenthesis preceded by the sign plus, without affecting the value of the expression.

Thus, $a+(b-c)=a+b-c$; $6+(5-3)=6+5-3$, and $a+b-c+d=a+(b-c+d)$; $5+4-3+2=5+(4-3+2)$.

2d. *Where the sign minus precedes a vinculum, it may be omitted if the signs of all the terms within it be changed.*

For the *minus* indicates subtraction, which is effected by changing the signs of all the terms of the quantity to be subtracted.

Thus, $a-(b-c)=a-b+c$.
$a-(x-y+z)=a-x+y-z$.

Sometimes several brackets, or vinculums, are employed in the same expression, all of which may be removed.

$$
\begin{aligned}
\text{Thus,}\quad & a-\{a+b-[a+b-c-(a-b+c)]\}, \\
&=a-\{a+b-[a+b-c-a+b-c]\}, \\
&=a-\{a+b-a-b+c+a-b+c\}, \\
&=a-a-b+a+b-c-a+b-c=b-2c.
\end{aligned}
$$

3d. *Any quantity may be inclosed in a parenthesis preceded by the sign minus, provided the signs of all the inclosed terms be changed.*

This is evident from the preceding principle.

Thus, $a-b+c=a-(b-c)=c-(b-a)$.

This principle often enables us to express the same quantity under several different forms.

$$\text{Thus,}\quad a-b+c+d=a-\{b-c-d\},$$
$$=a-\{b-(c+d)\}.$$

Simplify, as much as possible, the following expressions:

1. $(1-2x+3x^2)+(3+2x-x^2)$. Ans. $4+2x^2$.
2. $(a-b-c)+(b+c-d)+(d-e+f)+(e-f-g)$. Ans $a-g$.
3. $3(x^2+y^2)-\{(x^2+2xy+y^2)-(2xy-x^2-y^2)\}$. Ans. x^2+y^2.
4. $a-(x-a)-\{x-(a-x)\}$. Ans. $3a-3x$.
5. $1-\{1-[1-(1-x)]\}$. Ans. x.

OBSERVATIONS ON ADDITION AND SUBTRACTION.

47. All quantities are to be regarded as positive, unless, for some special reason, they are otherwise designated. Negative quantities are always, in some particular respect, the *opposite* of positive quantities. Thus:

If a merchant's *gains* are positive, his *losses* are negative; if latitude *north* of the equator is +, that *south* is —; if distance to the *right* of a certain line is +, that to the *left* is —; if time *after* a certain hour is +, time *before* that hour is —; if motion in one direction be +, motion in an opposite direction is —; and so on.

48. This relation of the signs gives rise to some important particulars.

1st. *The addition, to any quantity, of a negative number, produces a* LESS *result than adding zero.*

Thus,							
	10	10	10	10	10	10	10
	3	2	1	0	—1	—2	—3
	13	12	11	10	9	8	7

It will also be seen, from this illustration, that adding a negative number produces the same result as subtracting an equal positive number.

2d. *The subtraction of a negative quantity produces a greater result than subtracting zero.*

Thus,	10	10	10	10	10	10	10
	3	2	1	0	—1	—2	—3
	7	8	9	10	11	12	13

Here, subtracting a negative number produces the same result as adding an equal positive number.

49. When two negative quantities are *considered algebraically*, that is called the *least* which contains the greatest number of units; thus, —3 is said to be less than —2. But, that which contains the greatest number of units is said to be *numerically* the greatest; thus, —3 is numerically greater than —2.

50. The sum of two positive quantities is always *greater* than either of them. Thus, $+5+3=+8$.

The sum of two negative quantities, algebraically considered, is *less* than either of them. Thus, $-5-3=-8$.

The sum of a positive and negative quantity is always less than the positive quantity. Thus, $+5-3=+2$.

51. The difference of two positive quantities, as in arithmetic, is always less than the greater quantity. Thus, $2a$ from $5a$ leaves $3a$, or $5a-(+2a)=+3a$.

The difference of two negative quantities is always greater, algebraically considered, than the minuend. Thus, $-2a$ from $-5a$ leaves $-3a$, or $-5a-(-2a)=-3a$.

The difference between a positive and a negative quantity, found by subtracting the latter from the former, is always greater than either of them. Thus, $2a-(-a)=3a$

1. The latitude of A is 10° N. (+); the latitude of B is 5° S. (—); what their difference of latitude?

Ans. 15°.

2. At 7 A. M., a thermometer stood at —9°; at 2 P. M., at +15°; what was the change of temperature?

Ans. 24°.

MULTIPLICATION.

52. Multiplication, in Algebra, is the process of taking one algebraic quantity as many times as there are units in another.

The quantity to be multiplied is called the *multiplicand;* the quantity by which we multiply, the *multiplier;* and the result, the *product.*

The multiplicand and multiplier are called *factors.*

53. In explanation of the subject of algebraic multiplication, we begin with the following

Preliminary Principle.—*The product of two factors is the same, whichever be made the multiplier.*

To prove this, suppose we have a sash containing a vertical and b horizontal rows

Since there are a vertical rows and b panes in each row, the whole number of panes will be represented by b taken a times; that is, by ab, or by a taken b times; that is, by ba. Hence, ab is equal to ba.

In a similar manner, it may be shown that

The product of three, or of any number of factors, is the same, in whatever order they are taken.

Thus, $a\times b\times c=abc$, cab, bac, or cba, and $2\times3\times4=4\times2\times3=3\times2\times4=4\times3\times2$; the product in each case being 24. Also, $ac\times6=6ac$, or $6ca$; and so on.

It also follows from this principle, that

When either of the factors of a product is multiplied, the product itself is multiplied.

Thus, 2×3, multiplied by 5, may be written $5\times2\times3$, or $5\times3\times2$; that is, 10×3, or 15×2, either of which is equal to 30.

REMARK.—The distinction between the multiplication of numbers and of *factors* should be carefully noticed. Thus, $32\times2=64$, but 3×2 multiplied by 2 equals 6×2, or 3×4.

54. In multiplication there are four things to be considered in relation to each term, viz. the *coëfficient;* the *literal part;* the *exponent;* and the *sign.*

55. Of the Coefficient and Literal Part.—1. Let it be required to find the product of $2ac$ by $3b$.

To indicate the multiplication, we may write the product thus, $2ac\times3b$. But, by Art. 53, this is the same as $2\cdot3\times abc$, and $2\times3=6$; therefore, the product is $6abc$. Hence,	OPERATION. $2ac$ $3b$ $6abc$ product.

Rule of the Coefficients.—*Multiply together the coëfficients of the factors for the coëfficient of the product.*

Rule for the Literal Part.—*Annex to the coëfficient all the letters of the factors in alphabetical order.*

2. $3ac\times5b=$	$15abc$.	4. $5a\times4ax=$	$20aax$.
3. $2am\times cn=$	$2acmn$.	5. $7cy\times3yz=$	$21cyyz$.

56. Of the Exponent.—To determine the rule of the exponents,

1. Let it be required to find the product of $2a^2$ by $3a^3$.

Since $2a^2=2aa$, and $3a^3=3aaa$, the product will be $2aa\times3aaa$, or $6aaaaa$, which, for the sake of brevity, is written $6a^5$. Hence, we have the following	OPERATION. $2a^2=2aa$ $3a^3=3aaa$ $6a^5=6aaaaa$

Rule of the Exponents.—*Add the exponents of any letter in the factors for its exponent in the product.*

2. $ab\times a=$	a^2b	5. $a^m\times a^n=$	a^{m+n}.
3. $x^2y\times xy=$	x^3y^2	6. $c^{n+1}\times c^{n-1}=$	c^{2n}.
4. $a^3x^2z\times axz^2=$	$a^4x^3z^3$	7. $x^{m+p}\times x^{n-p}=$	x^{m+n}.

57. From the two preceding articles, we derive the following

GENERAL RULE FOR MULTIPLYING ONE POSITIVE MONOMIAL BY ANOTHER.

1. *Multiply the coëfficients for the coëfficient of the product.*
2. *Annex all the letters found in both factors.*
3. *When the same letter occurs in both, add its exponents.*

1. Multiply bc by z. . . . . Ans. bcz.
2. Multiply $3ax$ by by. . . . . . . Ans. $3abxy$.
3. Multiply $4am$ by $3bn$. . . . . Ans. $12abmn$.
4. Multiply $5a^2x$ by $7ax^3y$. . . . . Ans. $35a^3x^4y$.
5. Multiply $3a^mx^n$ by $9a^nx^m$. . . Ans. $27a^{m+n}x^{m+n}$.

58.—1. Required to find the product of $a+b$ by m.

Here, the sum of the units in a and b is to be taken m times. The units in a taken m times $=ma$, and the units in b taken m times $=mb$; hence, $a+b$ taken m times $=ma+mb$. Hence, when the signs are positive, we have the following

OPERATION.

$$\begin{array}{l} a+b \\ m \\ \hline ma+mb \end{array}$$

Rule for Multiplying a Polynomial by a Monomial.—*Multiply each term of the multiplicand by the multiplier.*

2. Multiply $x+y$ by n. . . . . Ans. $nx+ny$.
3. ax^2+cz by $3ac$. . . . . . . Ans. $3a^2cx^2+3ac^2z$.
4. $2a^2+3b^2$ by $5ab$. . . . . Ans. $10a^3b+15ab^3$.
5. $mx+ny+vz$ by m^2n. Ans. $m^3nx+m^2n^2y+m^2nvz$.

59.—1. Required to find the product of $a+b$ by $m+n$.

Here, $a+b$ is to be taken as many times as there are units in $m+n$, which is evidently as many times as there are units in m, plus as many times as there are units in n.

Thus,
$$\begin{array}{ll} a+b & \\ m+n & \\ \hline ma+mb= & \text{multiplicand taken } m \text{ times.} \\ \qquad na+nb= & \text{multiplicand taken } n \text{ times.} \\ \hline ma+mb+na+nb= & \text{multiplicand taken } (m+n) \text{ times.} \end{array}$$

Hence, when the signs are positive, we have the following

Rule for Multiplying one Polynomial by Another.—*Multiply each term of the multiplicand by each term of the multiplier, and add the products.*

2. Multiply $x+y$ by $a+c$. . . . Ans. $ax+ay+cx+cy$.
3. $2x+3z$ by $3x+2z$. . . . . Ans. $6x^2+13xz+6z^2$.
4. $2a+c$ by $a+2c$. . . . . . Ans. $2a^2+5ac+2c^2$.
5. x^2+xy+y^2 by $x+y$. . Ans. $x^3+2x^2y+2xy^2+y^3$.
6. $a^2+2ab+b^2$ by $a+b$. . Ans. $a^3+3a^2b+3ab^2+b^3$.

60. Of the Signs.—In the preceding article it was assumed that the product of two positive quantities is positive. The general rule for this, and the other cases which may arise in algebraic multiplication, may be deduced, as follows:

1st. Let it be required to find the product of $+b$ by a.

The quantity b, taken once, is $+b$; taken twice, is $+2b$; taken 3 times, is $+3b$; and hence, taken a times, it is $+ab$.

Hence, the product of two positive quantities is positive; or, more briefly expressed, *plus* multiplied by *plus* gives *plus*.

2d. Let it be required to find the product of $-b$ by a.

The quantity $-b$, taken once, is $-b$; taken twice, is $-2b$; taken 3 times, is $-3b$; and hence, taken a times, is $-ab$.

Hence, a *negative* quantity, multiplied by a *positive* quantity, gives a *negative* product; or, more briefly, *minus* multiplied by *plus* gives *minus*.

3d. Let it be required to multiply b by $-a$.

Since $b\times+a$ implies that b is to be *added* a times to 0, $b\times-a$ must indicate (Art. 47) that b is to be *subtracted* a times from 0. Subtracted once, it is $-b$; subtracted twice, $-2b$; and so on. Hence, subtracted a times, it is $-ab$.

Therefore, a *positive* multiplied by a *negative* quantity, gives a *negative* product; or, *plus* multiplied by *minus* gives *minus*.

4th. Let it be required to multiply $-b$ by $-a$.

Reasoning as above, $-b$ *subtracted* a times from 0, gives $+ab$. Hence, the product of two negative quantities is positive; or, more briefly, *minus* multiplied by *minus* gives *plus*.

NOTE.—The following proof of the 3d and 4th cases is generally regarded as more satisfactory than the preceding.

Let it be required to find the product of $c-d$ by $a-b$.

Here it is required to take $c-d$ as many times as there are units in $a-b$. This will be done by taking $c-d$ as many times as there are units in a, and then subtracting, from this product, $c-d$ taken as many times as there are units in b.

Thus,
$$\begin{array}{l}
\quad c-d \\
\quad \underline{a-b} \\
ac-ad=c-d \text{ taken } a \text{ times.} \\
\underline{\qquad bc-bd=c-d} \text{ taken } b \text{ times.} \\
ac-ad-bc+bd. \text{ By subtraction, } =c-d \text{ taken } a-b \text{ times.}
\end{array}$$

The final result, in the terms, $-bc$ and $+bd$, is what it would have been if we had *added* the partial products, assuming that $+c$ multiplied by $-b$ gives $-bc$, and that $-d$ multiplied by $-b$ gives $+bd$. As we know the *result* to be correct, we infer that the *assumption* would be correct, viz.: that *plus* by *minus* gives *minus*, and *minus* by *minus* gives *plus*.

From the above, we derive the following

General Rule for the Signs.—*The product of like signs gives plus, and of unlike signs, minus.*

GENERAL RULE FOR THE MULTIPLICATION OF ALGEBRAIC QUANTITIES.

61.—1. *Multiply every term of the multiplicand by each term of the multiplier, observing the rules for the coëfficients, the exponents, and the signs.*

2. *Add the several partial products together.*

NUMERICAL EXAMPLES TO VERIFY THE RULE OF THE SIGNS.

1. Multiply $7-4$ by 5. Ans. $35-20=15=3\times 5$.
2. $8+3$ by $6-4$. Ans. $48-14-12=22=11\times 2$.

GENERAL EXAMPLES.

1. Multiply $4a^2-3ac+2$ by $5ax$.
Ans. $20a^3x-15a^2cx+10ax$.
2. $5a-2ab+10$ by $-9ab$.
Ans. $-45a^2b+18a^2b^2-90ab$.
3. $2x+3z$ by $2x-3z$. Ans. $4x^2-9z^2$.
4. $4a^2-6a+9$ by $2a+3$. Ans. $8a^3+27$.
5 $a-b+c-d$ by $a+b-c-d$.
Ans. $a^2-b^2-c^2+d^2-2ad+2bc$.
6. $x^3+y^2+z^2$ by x^2+y^2.
Ans. $x^5+x^2y^2+x^2z^2+x^3y^2+y^4+y^2z^2$.
7. $a^3+3a^2b+3ab^2+b^3$ by $a^3-3a^2b+3ab^2-b^3$.
Ans. $a^6-3a^4b^2+3a^2b^4-b^6$.
8 $12x^3-8x^2y+15xy^2-10y^3$ by $3x+2y$.
Ans. $36x^4+29x^2y^2-20y^4$.
9. a^2+ax+x^2 by a^2-ax+x^2. Ans. $a^4+a^2x^2+x^4$.
10. $a^2+2ab+2b^2$ by $a^2-2ab+2b^2$. Ans. a^4+4b^4.
11. $1+x+x^2+x^3+x^4$ by $1-x$. Ans. $1-x^5$.
12. $27x^3+9x^2y+3xy^2+y^3$ by $3x-y$. Ans. $81x^4-y^4$.
13. $a^3+2a^2b+2ab^2+b^3$ by $a^3-2a^2b+2ab^2-b^3$.
Ans. a^6-b^6.
14. $x^4-x^3+x^2-x+1$ by x^2+x-1.
Ans. $x^6-x^4+x^3-x^2+2x-1$.
15. $1+x+x^4+x^5$ by $1-x+x^2-x^3$ Ans. $1-x^8$.
16. Multiply together $x-3$, $x+4$, $x-5$, and $x+6$.
Ans. $x^4+2x^3-41x^2-42x+360$.
17. $a+b$, $a-b$, a^2+ab+b^2, and a^2-ab+b^2.
Ans. a^6-b^6.

62. Multiplication by Detached Coefficients.—In the multiplication of polynomials, it is evident that the coëfficients of the product depend on the coëfficients of the factors, and not upon the literal parts of the terms.

Hence, by *detaching* the coëfficients of the factors from the literal parts, and multiplying them together, we shall obtain the coëfficients of the product. If to these coëfficients, the proper letters are then annexed, the whole product will be obtained. This method is applicable where the powers of the same letter increase or decrease regularly.

1. Multiply $a^2-2ab+b^2$ by $a+b$.

OPERATION.

$$\begin{array}{r} 1-2+1 \\ 1+1 \\ \hline 1-2+1 \\ +1-2+1 \\ \hline 1-1-1+1 \end{array}$$

After finding the coëfficients, it is obvious that a^3 will be the first term, and b^3 the last term; hence, the entire product is $a^3-a^2b-ab^2+b^3$.

2. Multiply $a^3-3a^2b+b^3$ by a^2-b^2.

OPERATION

$$\begin{array}{r} 1-3+0+1 \\ 1+0-1 \\ \hline 1-3+0+1 \\ -1+3-0-1 \\ \hline 1-3-1+4-0-1 \end{array}$$

In this example, supposing the powers of a to decrease regularly toward the left, it is obvious that there is a term wanting in each factor. These must be supplied by 0. The entire product is $a^5-3a^4b-a^3b^2+4a^2b^3-b^5$.

3. Multiply $m^3+m^2n+mn^2+n^3$ by $m-n$. Ans. m^4-n^4.

4. Multiply $1+2z+3z^2+4z^3+5z^4$ by $1-z$.

Ans. $1+z+z^2+z^3+z^4-5z^5$.

By this method, let the general examples, Art. 61, from 7 to 14 inclusive, be solved.

REMARKS ON ALGEBRAIC MULTIPLICATION.

63. The *degree* of the product of any two monomials is equal to the *sum* of the degrees of the multiplicand and multiplier. Thus, $2a^2b$, which is of the 3d degree, multiplied by $3ab^3$ of the 4th degree, gives $6a^3b^4$, which is of the 7th degree.

This is also true of two polynomials; as an illustration of which, see Example 7, Art. 61.

64. In the multiplication of two polynomials, when the partial products do not contain *similar terms*, if there be m terms in the multiplicand, and n terms in the multiplier, the number of terms in the product will be $m\times n$. Thus, in Example 6, Art. 61, there are 3 terms in the multiplicand, 2 in the multiplier, and $3\times2=6$ in the product.

65. If the partial products contain *similar terms*, the number of terms in the reduced product will evidently be less than $m\times n$; see Examples 7 to 18 inclusive, Art. 61.

66. When the multiplication of two polynomials, *indicated* by a parenthesis, as $(m+n)(p-q)$, is actually performed, the expression is said to be *expanded*, or *developed*.

DIVISION.

67. Division, in Algebra, is the process of finding how many times one algebraic quantity is contained in another.

Or, having the product of two factors, and one of them given, Division teaches the method of finding the other.

The quantity by which we divide is called the *divisor;* the quantity to be divided, the *dividend;* the result of the operation, the *quotient.*

68. In division, as in multiplication, there are four things to be considered, viz., the *sign;* the *coëfficient;* the *exponent;* and the *literal part.*

69. To ascertain the rule of the signs.

$$\text{Since, }\left.\begin{array}{r} +a\times+b=+ab \\ -a\times+b=-ab \\ +a\times-b=-ab \\ -a\times-b=+ab \end{array}\right\}\text{ therefore, }\left\{\begin{array}{l} +ab\div+b=+a \\ -ab\div+b=-a \\ +ab\div-b=-a \\ -ab\div-b=+a \end{array}\right.$$

From the foregoing illustration, we derive the following

Rule of the Signs.—*Like signs in the divisor and dividend give plus in the quotient; unlike signs give minus.*

70. The *rule of the coëfficients*, the *rule of the exponents*, and the *rule of the literal part*, may all be derived from the solution of a single example.

Required to find how often $2a^2$ is contained in $6a^5b$.

$$\frac{6a^5b}{2a^2}=\frac{6}{2}a^{5-2}b=3a^3b.$$

Since division is the reverse of multiplication, the quotient multiplied by the divisor, must produce the dividend; hence, to obtain this quotient, it is obvious,

1st. That the coëfficient of the quotient must be such a number, that when multiplied by 2 the product shall be 6; therefore, to obtain it, we divide 6 by 2. Hence, the

Rule of the Coefficients.—*Divide the coëfficient of the dividend by the coëfficient of the divisor.*

2d. The exponent of a in the quotient must be such a number, that when 2, the exponent of a in the divisor, is added to it, the sum shall be 5; that is, it must be 3, or 5—2. Hence, the

Rule of the Exponents.—*Subtract the exponent of any letter in the divisor from the exponent of the same letter in the dividend for its exponent in the quotient.*

3d. The letter b, which is a factor of the dividend, but not of the divisor, must be in the quotient. Hence, the

Rule of the Literal Part.—*Write, in the quotient, every letter found in the dividend, and not in the divisor.*

71. The preceding rules, taken together, give the following

GENERAL RULE FOR DIVIDING ONE MONOMIAL BY ANOTHER.

1. *Prefix the proper sign, on the principle that like signs give plus, and unlike signs give minus.*

2. *Divide the coëfficient of the dividend by that of the divisor.*

3. *Subtract the exponent of the divisor from that of the dividend, when the same letter or letters occur in both.*

4. *Annex any letter found in the dividend but not in the divisor.*

1. Divide $4a^5$ by $2a^2$ and by $-2a^2$. Ans. $2a^3$ and $-2a^3$.
2. $30a^4b^2$ by $5a^2b$. . . . . Ans. $6a^2b$.
3. $-28x^3y^7z^4$ by $-7xy^2z$. . . . . Ans. $4x^2y^5z^3$.
4. $-35a^2b^3c$ by $5ab^2$. . . . . . . Ans. $-7abc$.
5. $32xyz$ by $-8xy$. . . . . . Ans. $-4z$.
6. $42c^3m^2n$ by $-3cmn$. . . . Ans. $-14c^2m$.
7. x^{m+n} and x^{m-n} each by x^n. Ans. x^m and x^{m-2n}.
8. v^{m+n} by v^{m+p}. . . . . Ans. v^{n-p}.

NOTE.—In the following examples, the quantities included within the parenthesis are to be considered together, as a single quantity

9. Divide $(a+b)^3$ by $(a+b)^2$. . . . . Ans. $(a+b)$.
10. $(m-n)^5$ by $(m-n)^2$ . . . . . Ans. $(m-n)^3$.
11. $8(a-b)^3x^2y$ by $2(a-b)xy$. Ans. $4(a-b)^2x$.
12. $(a+bx^2)^{p+1}$ by $(a+bx^2)^{p-1}$ . . Ans. $(a+bx^2)^2$.

72. It is evident that one monomial can not be divided by another in the following cases:

1st. When the coëfficient of the dividend is not exactly divisible by the coëfficient of the divisor.

2d. When the same literal factor has a greater exponent in the divisor than in the dividend.

3d. When the divisor contains one or more literal factors not found in the dividend.

In each of these cases the division is to be indicated by a fraction. See Art. 119.

73. It has been shown, Art. 53, that any product is multiplied by multiplying either of its factors; hence, conversely, *any dividend will be divided by dividing either of its factors.*

Thus, $6\times9\div3=2\times9$; or, $6\times3=18$.

74. Division of Polynomials by Monomials.—In multiplying a polynomial by a monomial, we multiply each term of the multiplicand by the multiplier. Hence, conversely, we have the following

RULE FOR DIVIDING A POLYNOMIAL BY A MONOMIAL.

Divide each term of the dividend by the divisor, according to the rule for the division of monomials.

NOTE.—Place the divisor on the left, as in arithmetic.

1. Divide a^2+ab by a. . . . . Ans. $a+b$.
2. $3xy+2x^2y$ by $-xy$. . . . . . . Ans. $-3-2x$.
3. $10a^2z-15z^2-25z$ by $5z$. . . . Ans. $2a^2-3z-5$.
4. $3ab+12abx-9a^2b$ by $-3ab$. Ans. $-1-4x+3a$.
5. $5x^3y^3-40a^2x^2y^2+25a^4xy$ by $-5xy$.
   Ans. $-x^2y^2+8a^2xy-5a^4$.
6. $4abc-24ab^2-32abd$ by $-4ab$.
   Ans. $-c+6b+8d$.
7. $a^mb^3+a^{m+1}b^2+a^{n-2}b$ by ab.
   Ans. $a^{m-1}b^2+a^mb+a^{n-3}$.
8. $3a(x+y)+c^2(x+y)^2$ by $x+y$. Ans. $3a+c^2(x+y)$.
9. $(b+c)(b-c)^2-(b-c)(b+c)^2$ by $(b+c)(b-c)$.
   Ans. $(b-c)-(b+c)=-2c$.
10. $b^2c(m+n)-bc^2(m+n)$ by $bc(m+n)$ Ans. $b-c$.

DIVISION OF ONE POLYNOMIAL BY ANOTHER.

75. To deduce a rule for the division of polynomials, we shall first form a product, and then reverse the operation.

Multiplication, or formation of a product.

$$\begin{array}{l}
a^3-5a^2b \\
\underline{a^2+2a\,b-b^2} \\
a^5-5a^4b \\
\quad +2a^4b-10a^3b^2 \\
\qquad\qquad\quad \underline{-a^3b^2+5a^2b^3} \\
a^5-3a^4b-11a^3b^2+5a^2b^3
\end{array}$$

Division, or decomposition of a product.

$$\begin{array}{ll|l}
 & a^5-3a^4b-11a^3b^2+5a^2b^3 & a^3-5a^2b \\
 & \underline{a^5-5a^4b} & \overline{a^2+2ab-b^2} \\
\text{1st R.} & +2a^4b-11a^3b^2+5a^2b^3 & \quad\text{Quotient.} \\
 & \underline{+2a^4b-10a^3b^2} & \\
\text{2d Remainder,} & -a^3b^2+5a^2b^3 & \\
 & \underline{-a^3b^2+5a^2b^3} & \\
\text{3d Remainder,} & \qquad 0 &
\end{array}$$

The dividend, or product, and the divisor, being given, (Art. 67), it is now required to find the quotient, or the other factor.

This dividend has been formed by multiplying the divisor by the several terms of the quotient, and adding the partial products together. These several unknown terms, constituting the quotient, we are now to find.

Arranging the dividend and divisor according to the decreasing powers of the letter a, it is plain that the division of a^5, the first term of the dividend, by a^3, the first term of the divisor, will give a^2, the first term of the quotient.

If we subtract from the dividend a^5-5a^4b, which is the product of the divisor a^3-5a^2b by a^2, the first term of the quotient, the *remainder* $+2a^4b-11a^3b^2+5a^2b^3$, will be the product of the divisor by the other terms of the quotient.

The first term $+2a^4b$ of the 1st remainder, is the product of the 1st term a^3 of the divisor by the 1st of the remaining unknown terms of the quotient; therefore, we shall obtain the 2d term of the required quotient, by dividing $+2a^4b$ by a^3; this gives $+2ab$.

Multiplying the divisor by $+2ab$, and subtracting the product, we have a 2d remainder, which is the product of the divisor by the remaining term or terms of the quotient; hence, the division of the 1st term $-a^3b^2$ of this 2d remainder, by the 1st term a^3 of the divisor, must give the 3d term of the quotient, which is found to be $-b^2$.

The *remainder* zero, shows that the quotient $a^2+2ab-b^2$ is exact, since the subtraction of the three partial products has exhausted the dividend.

It is immaterial whether the divisor be placed on the right or left of the dividend; by placing it on the right, it is more easily multiplied by the respective terms of the quotient.

76. From the above, we derive the following

RULE FOR THE DIVISION OF ONE POLYNOMIAL BY ANOTHER.

1. *Arrange the dividend and divisor with reference to a certain letter.*

2. *Divide the first term of the dividend by the first term of the divisor, for the first term of the quotient. Multiply the divisor by this term, and subtract the product from the dividend.*

3. *Divide the first term of the remainder by the first term of the divisor, for the second term of the quotient. Multiply the divisor by this term, and subtract the product from the last remainder.*

4. *Proceed in the same manner, and if the final remainder is 0, the division is said to be exact.*

1. Divide $15x^2+16xy-15y^2$ by $5x-3y$.

OPERATION.

$$\begin{array}{l|l}
15x^2+16xy-15y^2 & 5x-3y \\
\underline{15x^2-\ \ 9xy} & 3x+5y, \text{ Quotient.} \\
\qquad +25xy-15y^2 & \\
\qquad \underline{+25xy-15y^2} &
\end{array}$$

2. Divide m^2-n^2 by $m+n$.

OPERATION.

$$\begin{array}{l|l}
m^2-n^2 & m+n \\
\underline{m^2+mn} & m-n, \text{ Quotient.} \\
\quad -mn-n^2 & \\
\quad \underline{-mn-n^2} &
\end{array}$$

3. Divide x^3+y^3 by $x+y$.

OPERATION.

$$\begin{array}{l|l}
x^3+y^3 & x+y \\
\underline{x^3+x^2y} & x^2-xy+y^2, \text{ Quot.} \\
\quad -x^2y+\ y^3 & \\
\quad \underline{-x^2y-xy^2} & \\
\qquad\qquad +xy^2+y^3 & \\
\qquad\qquad \underline{xy^2+y^3} &
\end{array}$$

4. Divide $7x^2y+5xy^2+2x^3+y^3$ by $3xy+x^2+y^2$.

Arranging the divisor and dividend with reference to x, we have the following:

OPERATION.

$$\begin{array}{l|l}
2x^3+7x^2y+5xy^2+y^3 & x^2+3xy+y^2 \\
\underline{2x^3+6x^2y+2xy^2} & 2x+y, \text{ Quotient.} \\
\qquad x^2y+3xy^2+y^3 & \\
\qquad \underline{x^2y+3xy^2+y^3} &
\end{array}$$

5. Divide $x^2+x^3-7x^4+5x^5$ by $x-x^2$.

Division performed, by arranging both quantities according to the *ascending* powers of x.

$$\begin{array}{l|l}
x^2+x^3-7x^4+5x^5 & x-x^2 \\
\underline{x^2-x^3} & x+2x^2-5x^3, \\
\quad 2x^3-7x^4 & \text{Quotient.} \\
\quad \underline{2x^3-2x^4} & \\
\qquad -5x^4+5x^5 & \\
\qquad \underline{-5x^4+5x^5} &
\end{array}$$

Division performed, by arranging both quantities according to the *descending* powers of x.

$$\begin{array}{l|l}
5x^5-7x^4+x^3+x^2 & -x^2+x \\
\underline{5x^5-5x^4} & -5x^3+2x^2+x, \\
\quad -2x^4+x^3 & \text{Quotient.} \\
\quad \underline{-2x^4+2x^3} & \\
\qquad -x^3+x^2 & \\
\qquad \underline{-x^3+x^2} &
\end{array}$$

The two quotients above are the same, but differently arranged.

6. Divide $6x^2+5xy-4y^2$ by $3x+4y$. Ans. $2x-y$.

7. $x^3-40x-63$ by $x-7$. Ans. x^2+7x+9.

8. $3h^5+16h^4k-33h^3k^2+14h^2k^3$ by h^2+7hk.
Ans. $3h^3-5h^2k+2hk^2$.

9. a^5-243 by $a-3$. A. $a^4+3a^3+9a^2+27a+81$.

10. $x^6-2a^3x^3+a^6$ by $x^2-2ax+a^2$
Ans. $x^4+2ax^3+3a^2x^2+2a^3x+a^4$.

11. $1-6x^5+5x^6$ by $1-2x+x^2$.
Ans. $1+2x+3x^2+4x^3+5x^4$.

12. $p^2+pq+2pr-2q^2+7qr-3r^2$ by $p-q+3r$.
Ans. $p+2q-r$.

13. $4x^5+4x-x^3$ by $3x+2x^2+2$.
Ans. $2x^3-3x^2+2x$.

14. x^6-a^6 by $x^3+2ax^2+2a^2x+a^3$.
Ans. $x^3-2ax^2+2a^2x-a^3$.

15. Divide $m^2+2mp-n^2-2nq+p^2-q^2$ by $m-n+p-q$.
Ans. $m+n+p+q$.

16. $a^3+b^3+c^3-3abc$ by $a+b+c$.
Ans. $a^2+b^2+c^2-ab-ac-bc$.

17. $x^{m+n}+x^ny^n+x^my^m+y^{m+n}$ by x^n+y^m. A. x^m+y^n.

18. $ax^3-(a^2+b)x^2+b^2$ by $ax-b$. Ans. x^2-ax-b.

19. $a^{2m}-3a^mc^n+2c^{2n}$ by a^m-c^n. Ans. a^m-2c^n.

20. $x^4+x^{-4}-x^2-x^{-2}$ by $x-x^{-1}$. Ans. x^3-x^{-3}.

21. $a^8+a^6b^2+a^4b^4+a^2b^6+b^8$ by $a^4+a^3b+a^2b^2+ab^3+b^4$.
Ans. $a^4-a^3b+a^2b^2-ab^3+b^4$.

22. $a^2+(a-1)x^2+(a-1)x^3+(a-1)x^4-x^5$ by $a-x$. Ans. $a+x+x^2+x^3+x^4$.

23. $1-9x^8-8x^9$ by $1+2x+x^2$.
Ans. $1-2x+3x^2-4x^3+5x^4-6x^5+7x^6-8x^7$.

24. $1+2x$ by $1-3x$ to 5 terms in the quotient.
Ans. $1+5x+15x^2+45x^3+135x^4+$ ete.

77. Division by Detached Coefficients.—Division may sometimes be conveniently performed by detaching the coëfficients, as explained in Art. 62. Thus,

1. Let it be required to divide $x^2+2xy+y^2$ by $x+y$.

$$\begin{array}{l|l}
1+2+1 & 1+1 \\
1+1 & \overline{1+1} \\
\hline
\quad +1+1 & \\
\quad +1+1 & \\
\hline
\end{array}$$

Hence, the coëfficients of the quotient are 1 and 1. Also, $x^2\div x=x$, and $y^2\div y=y$; therefore, the quotient is $1x+1y$, or $x+y$.

2. Divide $12a^4-26a^3b-8a^2b^2+10ab^3-8b^4$ by $3a^2-2ab+b^2$.

$$\begin{array}{l|l}
12-26-8+10-8 & 3-2+1 \\
12-8+4 & \overline{4-6-8} \\
\hline
\quad -18-12+10 & \\
\quad -18+12-6 & \\
\hline
\qquad -24+16-8 & \\
\qquad -26+16-8 & \\
\hline
\end{array}$$

Hence, the coëfficients of the quotient are 4—6—8. Also, $a^4\div a^2=a^2$, and $b^4\div b^2=b^2$; therefore, the quotient is $4a^2-6ab-8b^2$.

3. Divide a^3+x^3 by $a+x$.

$$\begin{array}{l|l}
1+0+0+1 & 1+1 \\
\underline{1+1} & \overline{1-1+1} \\
\quad -1 & a^2-ax+x^2, \text{ Quotient.} \\
\quad \underline{-1-1} & \\
\qquad +1+1 & \\
\qquad \underline{+1+1} &
\end{array}$$

4. Divide $m^5-5m^4n+10m^3n^2-10m^2n^3+5mn^4-n^5$ by $m^2-2mn+n^2$. Ans. $m^3-3m^2n+3mn^2-n^3$.

5. Divide $a^6-3a^4b^2+3a^2b^4-b^6$ by $a^3-3a^2b+3ab^2-b^3$. Ans. $a^3+3a^2b+3ab^2+b^3$.

Most of the examples in Art. 76 may be solved by this method.

II. ALGEBRAIC THEOREMS,

DERIVED FROM MULTIPLICATION AND DIVISION.

REMARK —One of the chief objects of Algebra is to establish certain general truths. The following theorems serve to show some of its most simple applications.

78. Theorem I.—*The square of the sum of two quantities is equal to the square of the first, plus twice the product of the first by the second, plus the square of the second.*

Let a represent one of the quantities and b the other.

Then, $a+b=$ their sum; and $(a+b)\times(a+b)$, or $(a+b)^2=$ the square of their sum. By multiplying, we obtain $a^2+2ab+b^2$, which proves the theorem.

$$\begin{array}{l}
a+b \\
\underline{a+b} \\
a^2+ab \\
\underline{\quad +ab+b^2} \\
a^2+2ab+b^2
\end{array}$$

APPLICATION.

1. $(2+5)^2=4+20+25=49$.
2. $(2m+3n)^2=4m^2+12mn+9n^2$.

3. $(ax+by)^2=a^2x^2+2abxy+b^2y^2$.
4. $(ax^2+3xz^3)^2=a^2x^4+6ax^3z^3+9x^2z^6$.

79. Theorem II.—*The square of the difference of two quantities is equal to the square of the first, minus twice the product of the first by the second, plus the square of the second.*

Let a represent one of the quantities, and b the other.

Then, $a-b=$ their difference; and $(a-b)\times(a-b)$, or $(a-b)^2=$ the square of their difference. By multiplying, we obtain $a^2-2ab+b^2$, which proves the theorem.

$$\begin{array}{l}
a-b \\
\underline{a-b} \\
a^2-ab \\
\underline{\quad\;\; -ab+b^2} \\
a^2-2ab+b^2
\end{array}$$

APPLICATION.

1. $(5-3)^2=25-30+9=4$.
2. $(2x-y)^2=4x^2-4xy+y^2$.
3. $(3x-5z)^2=9x^2-30xz+25z^2$.
4. $(az-3cx)^2=a^2z^2-6acxz+9c^2x^2$.

80. Theorem III.—*The product of the sum and difference of two quantities, is equal to the difference of their squares.*

Let a represent one of the quantities, and b the other.

Then, $a+b=$ their sum, and $a-b=$ their difference. Multiplying, we obtain a^2-b^2, which proves the theorem.

$$\begin{array}{l}
a+b \\
\underline{a-b} \\
a^2+ab \\
\underline{\quad\;\; -ab-b} \\
a^2-b^2
\end{array}$$

APPLICATION.

1. $(7+4)(7-4)=49-16=33=11\times3$.
2. $(2x+y)(2x-y)=4x^2-y^2$.
3. $(3a^2+4b^2)(3a^2-4b^2)=9a^4-16b^4$.
4. $(3ax+5by)(3ax-5by)=9a^2x^2-25b^2y^2$.

81. Theorem IV.—*Any factor may be transferred from one term of a fraction to another, if, at the same time, the sign of its exponent be changed.*

Take the fraction $\frac{ax^5}{bx^3}$. Since we may divide both terms by the same quantity without changing the value of the fraction, (RAY'S ARITHMETIC, 3d Book, Art. 136), divide first by x^3, and then by x^5, (Art. 70). Thus,

$$\frac{ax^5}{bx^3}=\frac{ax^2}{b}, \text{ and } \frac{ax^5}{bx^3}=\frac{a}{bx^{3-5}}=\frac{a}{bx^{-2}} \therefore \frac{ax^2}{b}=\frac{a}{bx^{-2}}.$$

In a similar manner, it may be shown that $\frac{a}{bx^2}=\frac{ax^{-2}}{b}$.

Also, $\frac{1}{x^2}=\frac{x^{-2}}{1}=x^{-2}$, and $x^m=\frac{1}{x^{-m}}$, from which it follows that,

The reciprocal of a quantity is equal to the same quantity with the sign of its exponent changed.

EXAMPLES.

1. $\frac{a^2b}{cd^2}=\frac{b}{a^{-2}cd^2}=\frac{bd^{-2}}{a^{-2}c}$

2. $a^m=\frac{1}{a^{-m}}$.

3. $\frac{a}{b^m}=ab^{-m}=\frac{b^{-m}}{a^{-1}}$.

4. $a^{m-n}=\frac{1}{a^{n-m}}$.

82. Theorem V.—*Any quantity, whose exponent is* 0, *is equal to unity.*

If we divide a^2 by a^2, and apply the rule for the exponents (Art. 70), we find $\frac{a^2}{a^2}=a^{2-2}=a^0$; but, since any quantity is contained in itself once, $\frac{a^2}{a^2}=1$; therefore, $a^0=1$.

Similarly, $\frac{x^m}{x^m}=x^{m-m}=x^0$. But $\frac{x^m}{x^m}=1$; therefore, $x^0=1$, which proves the theorem.

By this notation, we may preserve the trace of a letter, which has disappeared in division. Thus, $\frac{a^2b}{ab}=a^{2-1}b^{1-1}=a^1b^0=a$.

83. Theorem VI.—*The difference of the same power of two quantities is always divisible by the difference of the quantities.*

If we divide a^2-b^2, a^3-b^3, etc., successively by $a-b$, the quotients will be found, by trial, to follow a simple law, both as to the exponents and the signs. Thus,

$$(a^2-b^2)\div(a-b)=a+b;$$
$$(a^3-b^3)\div(a-b)=a^2+ab+b^2;$$
$$(a^4-b^4)\div(a-b)=a^3+a^2b+ab^2+b^3;$$
$$(a^5-b^5)\div(a-b)=a^4+a^3b+a^2b^2+ab^3+b^4, \text{ etc.}$$

The general and direct proof of this theorem is as follows:

Let us divide a^m-b^m by $a-b$.

$$\begin{array}{l|l}
a^m-b^m & a-b \\
\hline
\underline{a^m-a^{m-1}b} & a^{m-1}+\dfrac{b(a^{m-1}-b^{m-1})}{a-b}, \text{ Quot.} \\
\quad a^{m-1}b-b^m=b(a^{m-1}-b^{m-1}) &
\end{array}$$

In performing this division, we see that the first term of the quotient is a^{m-1}, and the first remainder, $b(a^{m-1}-b^{m-1})$.

The remainder consists of two factors, b and $a^{m-1}-b^{m-1}$. Now, *if* the second of these factors, viz., $a^{m-1}-b^{m-1}$, is divisible by $a-b$, then will the quantity a^m-b^m be divisible by $a-b$. That is,

If the difference of the same powers of two quantities is divisible by the difference of the quantities themselves, then will the difference of the next higher powers of the same quantities be divisible by the difference of the quantities.

But we have seen that a^2-b^2 is divisible by $a-b$; hence, a^3-b^3 is also divisible by $a-b$. Again, since a^3-b^3 is divisible by $a-b$, it follows that a^4-b^4 is divisible by it, and so on; which proves the theorem generally.

84. Lemma.—In proving the next two theorems, it is necessary to notice, that the *even* powers of a negative quantity are *positive*, and the *odd* powers *negative*. Thus,

$-a$, the 1st power of $-a$, is negative.
$-a\times-a=a^2$, the 2d power, is positive
$-a\times-a\times-a=-a^3$, the 3d power, is negative.
$-a\times-a\times-a\times-a=a^4$, the 4th power, is positive; and so on.

85. Theorem VII.—*The difference of the even powers of the same degree of two quantities, is always divisible by the sum of the quantities.*

If we take the quantities $a-b$ and a^m-b^m, and put $-c$ instead of b, $a-b$ will become $a-(-c)=a+c$; and when m is *even*, b^m will become c^m, and a^m-b^m will become $a^m-(+c^m)=a^m-c^m$: but a^m-b^m is always divisible by $a-b$;

Therefore, a^m-c^m is always divisible by $a+c$ when m is *even*, which is the theorem.

EXAMPLES.

1. $(a^2-b^2)\div(a+b)=a-b$.
2. $(a^4-b^4)\div(a+b)=a^3-a^2b+ab^2-b^3$.
3. $(a^6-b^6)\div(a+b)=a^5-a^4b+a^3b^2-a^2b^3+ab^4-b^5$.

86. Theorem VIII.—*The sum of the odd powers of the same degree of two quantities, is always divisible by the sum of the quantities.*

If we take the quantities $a-b$ and a^m-b^m, and put $-c$ instead of b, $a-b$ will become $a-(-c)=a+c$; and when m is *odd*, b^m will become $-c^m$, (Art. 84), and a^m-b^m will become $a^m-(-c^m)=a^m+c^m$: but a^m-b^m is always divisible by $a-b$;

Therefore, a^m+c^m is always divisible by $a+c$ when m is *odd*, which is the theorem.

EXAMPLES.

1. $(a^3+b^3)\div(a+b)=a^2-ab+b^2$.
2. $(a^5+b^5)\div(a+b)=a^4-a^3b+a^2b^2-ab^3+b^4$.
3. $(a^7+b^7)\div(a+b)=a^6-a^5b+a^4b^2-a^3b^3+a^2b^4-ab^5+b^6$.

By a method of proof similar to that employed in Theorem VI., it may be shown that the sum of two quantities of the same degree can *never* be divided by the difference of the quantities. Thus, $a+b$, a^2+b^2, a^3+b^3, a^4+b^4, etc., are not divisible by $a-b$.

When, in either of the last three theorems, a or b becomes unity, the form of the quotient will be obvious. Thus,

$$(a^5-1)\div(a-1)=a^4+a^3+a^2+a+1.$$
$$(1+a^5)\div(1+a)=1-a+a^2-a^3+a^4, \text{ etc.}$$

FACTORING.

87. The following summary of the principles of arithmetic should be remembered:

Proposition I.—*A factor of any number is a factor of any multiple of that number.*

Proposition II.—*A factor of two numbers is a factor of their sum.*

From these are inferred the following, and the converse of each:

1. Every number ending in 0, 2, 4, 6, or 8, is divisible by 2.

2. Every number is divisible by 4, when the number denoted by its two right hand digits is divisible by 4.

3. Every number ending in 0 or 5, is divisible by 5.

4. Every number ending with 0, 00, etc., is divisible by 10, 100, etc.

88. A **Divisor** or **Factor** of a quantity, is a quantity that will exactly divide it without a remainder. Thus, a is a factor or divisor of ab, and $a+x$ is a divisor or factor of a^2-x^2.

89. A **Prime Quantity** is one which is exactly divisible, only by itself and unity. Thus, x, y, and $x+z$, are prime quantities; while xy, and $ax+az$, are not prime.

90. Two quantities are said to be *prime to each other*, or *relatively prime*, when no quantity except unity will exactly divide them both. Thus, ab and cd are prime to each other.

91. A **Composite Quantity** is one which is the product of two or more factors, neither of which is unity. Thus, a^2-x^2 is a composite quantity, the factors being $a+x$ and $a-x$.

92. To separate a monomial into its prime factors,

Rule.—*Resolve the coëfficient into its prime factors; then, these with the literal factors of the monomials, will be the prime factors of the given quantity.*

1. Find the prime factors of $18ab^2$. Ans. $2\times3\times3\times a.b.b.$
2. Of $28x^2yz^3$. . . . Ans. $2\times2\times7\times x.x.y.z.z.z.$
3. Of $210ax^3yz^2$. . . Ans. $2\times3\times5\times7.a.x.x.x.y.z.z.$

93. To separate a polynomial into its factors, when one of them is a monomial,

Rule.—*Divide the given quantity by the greatest monomial that will exactly divide each of its terms. The divisor will be one factor, and the quotient the other.*

1. Separate into factors, $a+ax$. . . Ans. $a(1+x)$.
2. $xz+yz$. . . . . Ans. $z(x+y)$.
3. x^2y+xy^2. . . . . Ans. $xy(x+y)$.
4. $6ab^2+9a^2bc$. . . . . Ans. $3ab(2b+3ac.)$
5. $a^2bx^3y—ab^2xy^2+abcxyz^2$. Ans. $abxy(ax^2—by+cz^2)$.

94. To separate any binomial or trinomial which is the product of two or more polynomials, into its prime factors.

1st. Any trinomial can be separated into two binomial factors, when the extremes are squares and positive, and the middle term is twice the product of the square roots of the extreme terms.

The factors will be the sum or difference of the square roots of the extreme terms, according as the sign of the middle term is plus or minus. (See Arts 78, 79.)

$$\text{Thus, } a^2+2ab+b^2=(a+b)(a+b);$$
$$a^2—2ab+b^2=(a—b)(a—b).$$

2d. Any binomial, which is the difference of two squares, can be separated into factors, one of which is the sum and the other the difference of their roots. (See Art. 80.)

$$\text{Thus, } a^2—b^2=(a+b)(a—b).$$

3d. Any binomial which is the difference of the same powers of two quantities, can be separated into at least *two* factors, one of which is the difference of the two quantities. (See Art. 83.)

Thus, $x^3-y^3=(x-y)(x^2+xy+y^2)$.
Similarly, $x^5-y^5=(x-y)(x^4+x^3y+x^2y^2+xy^3+y^4)$.

4th. Any binomial which is the *difference of the even powers* of two quantities, higher than the second degree, can be separated into at least *three* factors. (See Art. 85.)

Thus, $a^4-b^4=(a^2+b^2)(a^2-b^2)=(a^2+b^2)(a+b)(a-b)$.

5th. Any binomial which is the *sum of the odd powers* of two quantities, can be separated into at least *two* factors, one of which is the sum of the quantities. (Art. 86.)

Thus, $a^3+b^3=(a+b)(a^2-ab+b^2)$

6th. The following examples of the factoring of binomials composed of the sum of like even powers of quantities may be verified either by multiplication or by division:

$$a^2+b^2=(a+\sqrt{2ab}+b)(a-\sqrt{2ab}+b).$$
$$a^4+b^4=(a^2+\sqrt{2}.ab+b^2)(a^2-\sqrt{2}.ab+b^2).$$
$$a^6+b^6=(a^2+b^2)(a^4-a^2b^2+b^4)=(a^2+b^2)(a^2+\sqrt{3}.ab+b^2)(a^2-\sqrt{3}.ab+b^2).$$
$$a^8+b^8=(a^4+\sqrt{2}.a^2b^2+b^4)(a^4-\sqrt{2}.a^2b^2+b^4).$$
$$a^{10}+b^{10}=(a^2+b^2)(a^8-a^6b^2+a^4b^4-a^2b^6+b^8).$$
$$a^{12}+b^{12}=(a^4+b^4)(a^8-a^4b^4+b^8).$$
$$a^{2m}+b^{2m}=(a^m+\sqrt{2}.a^{\frac{m}{2}}b^{\frac{m}{2}}+b^m)(a^m-\sqrt{2}.a^{\frac{m}{2}}b^{\frac{m}{2}}+b^m).$$
$$a^{3m}+b^{3m}=(a^m+b^m)(a^{2m}-a^mb^m+b^{2m}).$$

Separate the following into their simplest factors:

1. $c^2+2cd+d^2$.
2. $a^2x^4+2ax^2y+y^2$.
3. $25x^2y^4+20xy^2z+4z^2$.
4. $9x^4-6x^2z^2+z^4$.
5. $4m^2x^2-4mn^2x+n^4$.
6. x^2-z^2.
7. $9a^2x^4-25$.
8. $16-a^2b^4z^6$.
9. a^4-x^4.
10. z^3+1.
11. y^3-1.
12. $a^3x^3-b^3y^3$.
13. x^5+y^5.
14. x^6-y^6.

94.[a] To separate a quadratic trinomial into its factors.

A **Quadratic Trinomial** is of the form x^2+ax+b, in which the sign of the second term may be either plus or minus.

Such a quantity may be resolved into factors by *inspection*. Observe carefully the product resulting from the multiplication of two factors of the form $x+a$, and $x+b$. Thus, $x^2-5x+6=(x-2)(x-3)$, since the first term of each factor must be x, and the other terms, -2 and -3, must be such that their sum will be -5, and their product $+6$.

Trinomials to be decomposed into binomial factors.

1. x^2+3x+2. . . . . Ans. $(x+1)(x+2)$.
2. $x^2-8x+15$. . . . . Ans. $(x-3)(x-5)$.
3. x^2-x-2. . . . . Ans. $(x+1)(x-2)$.
4. x^2+x-12. . . . . Ans. $(x-3)(x+4)$.
5. x^2-x-12. . . . . Ans. $(x+3)(x-4)$.
6. $x^2+2x-35$ . . . . Ans. $(x-5)(x+7)$.

95. Examples to be resolved into factors, by first separating the monomial factor, and then applying Arts. 93 and 94.

Ex. 1. $ax^3y-axy^3=axy(x^2-y^2)=axy(x+y)(x-y)$.

2. $3ax^2+6axy+3ay^2$. . . . . Ans. $3a(x+y)(x+y)$.
3. $2cx^2-12cx+18c$. . . . . Ans. $2c(x-3)(x-3)$.
4. $3m^3n-3mn^3$. . . . . . Ans. $3mn(m+n)(m-n)$.
5. $2x^5y-2xy^5$. . . . Ans. $2xy(x^2+y^2)(x+y)(x-y)$.
6. $2x^2+6x-8$. . . . . . . Ans. $2(x+4)(x-1)$
7. $2x^3+4x^2-70x$. . . . . Ans. $2x(x+7)(x-5)$.

Solve the following, by first indicating the operations to be performed, and then canceling common factors.

8. Multiply $4x-12$ by $1-x^2$, and divide the product by $2+2x$.

$$\frac{(4x-12)(1-x^2)}{2+2x}=\frac{4(x-3)(1+x)(1-x)}{2(1+x)}=2(x-3)(1-x)=$$
$$2(4x-3-x^2)=8x-6-2x^2.$$

9. Multiply $x^2+2xy+y^2$ by $x-y$, and divide the product by x^2-y^2. Ans. $x+y$

10. Multiply together $1-c$, $1-c^2$, and $1+c^2$, and divide the product by $1-2c+c^2$. Ans. $1+c+c^2+c^3$.

11. Multiply x^3-x^2-30x by $x^2+11x+30$, and divide the product by the product of x^2-36 and $x^2+10x+25$.

Ans. x.

GREATEST COMMON DIVISOR.

96. A **Common Divisor,** or **Common Measure,** is any quantity that will exactly divide two or more quantities. Thus, ab is a common divisor of ab^2 and abx.

REMARK.—Two quantities often have more than one common divisor. Thus, a^2cx and $abdx$ have three common divisors, a, x, and ax.

97. The **Greatest Common Divisor,** or **Greatest Common Measure** of two quantities, is the greatest quantity that will exactly divide each of them. Thus, the greatest common divisor of $6a^2bx^2$ and $9a^3cxz$ is $3a^2x$.

98. Quantities that have a common divisor are said to be *commensurable;* and those that have no common divisor, *incommensurable.*

NOTE.—G.C.D. stands for greatest common divisor.

99. To find the G.C.D. of two or more monomials.

1. Let it be required to find the G.C.D. of the two monomials, $14a^3cx$ and $21a^2bx$.

By separating each quantity into its prime factors, we have $14a^3cx=7\times2\times aaacx$, and $21a^2bx=7\times3\times aabx$.

The only factors *common* to both these quantities, are 7, aa or a^2, and x; hence, both can be divided by either of these factors, or by their product, $7a^2x$, and by no other quantity; therefore, $7a^2x$ is their G.C.D. Hence.

TO FIND THE GREATEST COMMON DIVISOR OF TWO OR MORE MONOMIALS,

Rule.—1. *Resolve the quantities into their prime factors.*

2. *Multiply together those factors that are common to all the terms, for the greatest common divisor.*

2. Find the G.C.D. of $6a^2xy$, $9a^3x^3$, and $15a^4x^4y^3$.

OPERATION.

$$6a^2xy = 3\times 2a^2xy$$
$$9a^3x^3 = 3\times 3a^3x^3$$
$$15a^3x^4y^3 = 3\times 5a^4x^4y^3$$

Here, 3 is the only numerical factor, and a and x the only letters common to all the quantities. The least powers of a and x, are a^2 and x; hence, the G.C.D. is $3a^2x$.

Find the G.C.D. of the following quantities:

3. $15abc^2$, and $21b^2cd$. . . . . Ans. $3bc$.
4. $4a^3b$, $10a^3c$, and $14a^2bc$. . . . . . . Ans. $2a^2$.
5. $4ax^4y^3$, $20x^4y^2z$, and $12x^3y^3z^2$. . . . Ans. $4x^3y^2$.
6. $12a^2x^2z^2$, $18ax^3z^2$, $30a^2x^3z$, and $6ax^3z^2$. Ans. $6ax^2z$.

100. Previous to investigating the rule for finding the G.C.D. of two polynomials, it is necessary to introduce the following propositions:

Proposition I.—*A divisor of any quantity is also a divisor of any multiple of that quantity.*

Thus, if A will divide B, it will divide 2B, 3B, etc.

Proposition II.—*A divisor of two quantities is also a divisor of their sum or their difference.*

Thus, if A will divide B and C, it will divide B+C, or B−C. This is evident from Art. 74.

101. Let it be required to find the G.C.D. of two polynomials, A and B, of which A is the greater.

If we divide A by B, and there is no remainder, B is evidently the G.C.D., since it can have no divisor greater than itself.

$$\begin{array}{l} B)A(Q \\ \underline{BQ} \\ A-BQ=R, \text{ 1st. Rem.} \end{array}$$

Divide A by B, and call the quotient Q; then if there is a remainder R, it is evidently equal to A—BQ. If, now, there is any common divisor of A and B, it will also divide BQ (Prop. 1st) and A—BQ or R (Prop. 2d); or the common divisor must divide A, B, and R, and can not be greater than R.

$$\begin{array}{l} R)B(Q' \\ \underline{RQ'} \\ B-RQ'=R', \text{ 2d Rem.} \end{array}$$

$$A=BQ+R$$
$$B=RQ'+R'$$

Since the dividend is equal to the product of the divisor by the quotient, plus the remainder.

Now, if R will exactly divide B, it will also exactly divide BQ (Prop. 1st) and BQ+R (Prop. 2d). Consequently, it will divide A, since A=BQ+R, and will be the common divisor of the two polynomials A and B. It will also be the *greatest* common divisor, since no divisor of A, B, and R can be greater than R.

Suppose, however, that when we undertake to divide R into B, to ascertain if it will *exactly* divide it, we find that the quotient is Q′, with a remainder R′.

Now, reasoning as before, if R′ exactly divides R, it will also divide RQ′ (Prop. 1st) and also B (Prop. 2d), since B=RQ′+R′; and whatever exactly divides B and R, will also exactly divide A, since A=BQ+R; therefore, if R′ exactly divides R, it will exactly divide both A and B, and will be their common divisor. It will also be the *greatest* common divisor, since the greatest divisor of R′ is R′ itself.

By continuing to divide the last divisor by the last remainder, we may apply the same reasoning to every successive divisor and remainder; and when any division becomes exact, the last divisor will be the greatest common measure of A and B.

The same method of proof may be applied to numbers; for example, let A=120, and B=35.

102. When a remainder becomes unity, or does not contain the letter of arrangement, it is evident that there is no common divisor of the two quantities.

103. If either quantity contains a factor not found in the other, that factor may be canceled without affecting the common divisor. Thus, a is the G.C.D. of ax and ay, and will be, if we cancel x in ax, or y in ay.

104. We may multiply either quantity by a factor *not found* in the other, without changing the G.C.D. Thus, in the two quantities, ax and ay, if we multiply ax by m, or ay by n, the G.C.D. will still be a.

105. But if we multiply either quantity by a factor found in the other, we change the G.C.D. Thus, in the two quantities, ax and ay, if we multiply ay by x, or ax by y the G.C.D. becomes ax or ay.

106. From Art. 101, it is evident that the three preceding articles apply also to the successive remainders.

107. It is evident that any common factor of two quantities, must also be a factor of their G.C.D. Where such common factor is easily seen, we may set it aside, and find the G.C.D. of what remains.

Thus, take $55x$ and $15x$. Setting aside x, we find the greatest common measure of 55 and 15 to be 5. Annexing x, we have $5x$.

REMARK.—The illustrative examples, in the five articles above, are monomials, but the same principles obviously apply to polynomials.

We shall now show the application of these principles.

1. Find the G.C.D. of x^3-z^3 and $x^4-x^2z^2$.

Here the second quantity contains x^2 as a factor, but it is not a factor of the first; we may, therefore, cancel it (Art. 103), and the second quantity becomes x^2-z^2. Then divide the first by it.

After dividing, we find that z^2 is a factor of the remainder, but not of x^2-z^2, the next dividend. We, therefore, cancel it (Art. 103), and the second divisor becomes $x-z$. Then, dividing by this, we find there is no remainder; therefore, $x-z$ is the G.C.D.

OPERATION.

$$
\begin{array}{l|l}
x^3-z^3 & x^2-z^2 \\
x^3-xz^2 & x \\
\hline
xz^2-z^3 & \\
\text{or } (x-z)z^2 & \\
\end{array}
$$

$$
\begin{array}{l|l}
x^2-z^2 & x-z \\
x^2-xz & x+z \\
\hline
xz-z^2 & \\
xz-z^2 & \\
\hline
\end{array}
$$

2. Find the G.C.D. of $x^5+x^2z^3$ and $x^5-x^3z^2$.

The factor x^2 is common to both quantities; it is, therefore, a factor of the greatest divisor (Art. 107), and may be taken out and reserved. Doing this, the quantities become x^3+z^3 and x^3-xz^2. The second quantity still contains a common factor, x, which the first does not; canceling this, it becomes x^2-z^2. Then, proceeding as in the first example, we find that $x+z$ divides without a remainder; therefore, $x^2(x+z)$ is the required G.C.D.

OPERATION.

$$\begin{array}{l|l}
x^3+z^3 & x^2-z^2 \\
\underline{x^3-xz^2} & \underline{x} \\
xz^2+z^3 & \\
\text{or } (x+z)z^2 & \\
\\
x^2-z^2 & x+z \\
\underline{x^2+xz} & \underline{x-z} \\
-xz-z^2 & \\
\underline{-xz-z^2} &
\end{array}$$

3. Find the G.C.D. of $10a^2x^2-4a^2x-6a^2$, and $5bx^2-11bx+6b$.

By separating the monomial factors, we find

$$10a^2x^2-4a^2x-6a^2=2a^2(5x^2-2x-3),$$
$$\text{and } 5bx^2-11bx+6b=b(5x^2-11x+6).$$

The factors $2a^2$ and b have no common measure, and hence are not factors of the common divisor. We may, therefore, suppress them (Art. 103), and proceed to find the G.C.D. of the remaining quantities, which is found to be $x-1$.

OPERATION.

$$\begin{array}{l|l}
5x^2-11x+6 & 5x^2-2x-3 \\
\underline{5x^2-\ \ 2x-3} & \underline{1} \\
\quad -9x+9 & \\
\text{or } -9(x-1) & \\
\\
5x^2-2x-3 & x-1 \\
\underline{5x^2-5x} & \underline{5x+3} \\
\quad 3x-3 & \\
\quad \underline{3x-3} &
\end{array}$$

4. Find the G.C.D. of $4a^2-5ay+y^2$, and $3a^3-3a^2y+ay^2-y^3$.

In solving this example, it is necessary, in two instances, to multiply the dividend, that the coëfficient of the first term may be divisible by the first term of the divisor (Art. 104.).

OPERATION.

$$\begin{array}{r|l}
3a^3-3a^2y+ay^2-y^3 & 4a^2-5ay+y^2 \\
\underline{4} & \\
12a^3-12a^2y+4ay^2-4y^3 & \underline{3a+3y} \\
\underline{12a^3-15a^2y+3ay^2} & \\
3a^2y+\ ay^2-4y^3 & \\
\underline{4} & \\
12a^2y+\ 4ay^2-16y^3 &
\end{array}$$

[OVER.]

We find $19y^2$ is a factor of the first remainder, but not of the first divisor, and hence can not be a factor of the G.C.D.; it must, therefore, be suppressed. Hence,

$$\begin{array}{r} 12a^2y+4ay^2-16y^3 \quad \text{[BROUGHT OVER.]} \\ \underline{12a^2y-15ay^2+3y^3} \\ 19ay^2-19y^3 \\ \text{or } 19y^2(a-y) \end{array}$$

$$\begin{array}{l|l} 4a^2-5ay+y^2 & a-y \quad \text{G.C.D.} \\ \underline{4a^2-4ay} & \overline{4a-y} \\ \quad -ay+y^2 & \\ \quad \underline{-ay+y^2} & \end{array}$$

TO FIND THE GREATEST COMMON DIVISOR OF TWO POLYNOMIALS,

108. Rule.—1. *Divide the greater polynomial by the less, and if there is no remainder, the less quantity will be the divisor sought.*

2. *If there be a remainder, divide the first divisor by it, and continue to divide the last divisor by the last remainder, until a divisor is obtained which leaves no remainder; this will be the G.C.D. of the two given polynomials.*

NOTES.—1. When the highest power of the *leading* letter is the same in both, it is immaterial which of the quantities is made the dividend.

2. If both quantities contain a common factor, let it be set aside, as forming a factor of the common divisor, and proceed to find the G.C.D. of the remaining factors, as in Ex. 2.

3. If either quantity contains a factor not found in the other, it may be canceled before commencing the operation, as in Ex. 3.

4. Whenever it is necessary, the dividend may be multiplied by any quantity which will render the first term exactly divisible by the first term of the divisor, as in Ex. 4.

5. If, in any case, the remainder is unity, or does not contain the leading letter, there is no common divisor.

6. To find the G.C.D. of three or more quantities, first find the G.C.D. of two of them; then of that divisor and one of the other quantities, and so on. The last divisor thus found will be the G.C.D. sought.

7. Since the G.C.D. of any two quantities contains all the factors common to both, it may often be found most easily by separating the polynomials into factors. (Arts. 92 to 95.)

Find the G.C.D. in the following quantities:

1. $5x^2-2x-3$ and $5x^2-11x+6$. . . . Ans. $x-1$.
2. $9x^2-4$ and $9x^2-15x-14$. . . . . Ans. $3x+2$.
3. $a^2-ab-12b^2$ and $a^2+5ab+6b^2$. . . Ans. $a+3b$.
4. a^4-x^4 and $a^3+a^2x-ax^2-x^3$. . . . . Ans. a^2-x^2.
5. $x^3-5x^2+13x-9$ and x^3-2x^2+4x-3. Ans. $x-1$.
6. $21x^3-26x^2+8x$ and $6x^2-x-2$. . . Ans. $3x-2$.
7. x^4+2x^2+9 and $7x^3-11x^2+15x+9$. Ans. x^2-2x+3.
8. x^2+5x+4, x^2+2x-8, and $x^2+7x+12$. Ans. $x+4$.
9. $2b^3-10ab^2+8a^2b$ and $9a^4-3ab^3+3a^2b^2-9a^3b$. Ans. $a-b$.
10. $x^4+a^2x^2+a^4$ and $x^4+ax^3-a^3x-a^4$. Ans. x^2+ax+a^2.
11. $x^4-px^3+(q-1)x^2+px-q$ and $x^4-qx^3+(p-1)x^2+qx-p$. Ans. x^2-1.

LEAST COMMON MULTIPLE.

109. A **Multiple** of a quantity is any quantity that contains it exactly. Thus, 6 is a multiple of 2 or of 3; and ab is a multiple of a or of b; also, $a(b-c)$ is a multiple of a or $(b-c)$.

110. A **Common Multiple** of two or more quantities, is a quantity that contains either of them exactly. Thus, 12 is a common multiple of 2 and 3; and $20xy$, of $2x$ and $5y$.

111. The **Least Common Multiple** of two or more quantities, is the least quantity that will contain them exactly. Thus, 6 is the least common multiple of 2 and 3; $10xy$, of $2x$ and $5y$.

NOTE.—L.C.M. stands for least common multiple.

112. To find the L.C.M. of two or more quantities.

It is evident that the L.C.M. of two or more quantities *contains all the prime factors of each of the quantities once, and does not contain any prime factor besides.*

Thus, the L.C.M. of ab and bc must contain the factors a, b, c, and no other factor.

Assuming the principle above stated, let us find the L.C.M. of mx, nx, and m^2nz.

OPERATION.

$$\begin{array}{r|lll} m & mx & nx & m^2nz \\ \hline n & x & nx & mnz \\ \hline x & x & x & nz \\ \hline & 1 & 1 & mz \end{array}$$

$m\times n\times x\times mz=m^2nxz$

Arranging the quantities as in the margin, we see that m is a prime factor common to two of them. It must, therefore, even if found in only one of the quantities, be a factor of the L.C.M.; and as it can occur but once in the L.C.M., we cancel m in each of the quantities in which it is found, which is done by dividing by it. For the same reason we divide by n and by x.

We thus find that the L.C.M. must contain the factors m, n, and x; also, mz, otherwise it would not contain all the prime factors found in one of the quantities. Hence, $m\times n\times x\times mz=m^2nxz$, contains all the prime factors of the quantities once, and contains no other factor; it is, therefore, the required L.C.M. Hence,

TO FIND THE LEAST COMMON MULTIPLE OF TWO OR MORE QUANTITIES,

Rule.—1. *Arrange the quantities in a horizontal line, divide by any prime factor that will exactly divide two or more of them, and set the quotients and the undivided quantities in a line beneath.*

2. *Continue dividing as before, until no prime factor, except unity, will exactly divide two or more of the quantities.*

3. *Multiply the divisors and the quantities in the last line together, and the product will be the L.C.M. required.*

Or, *Separate the quantities into their prime factors; then, to form a product,* 1st, *take each factor once;* 2d, *if any factor occurs more than once, take it the greatest number of times it occurs in either of the quantities.*

113. Since the G.C.D. of two quantities contains all the factors common to both, *if we divide the product of two quantities by their G.C.D., the quotient will be their L.C.M.*

1. Find the L.C.M. of $6a^2$, $9ax^3$, and $24x^5$. Ans. $72a^2x^5$.
2. $32x^2y^2$, $40ax^5y$, $5a^2x(x-y)$. Ans. $160a^2x^5y^2(x-y)$.
3. $3x+6y$ and $2x^2-8y^2$. . . . . Ans. $6x^2-24y^2$.
4. a^3+x^3 and a^2-x^2. . . Ans. $a^4-a^3x+ax^3-x^4$.
5. $x-1$, x^2-1, $x-2$, and x^2-4. Ans. x^4-5x^2+4.
6. x^2-1, x^2+1, $(x-1)^2$, $(x+1)^2$, x^3-1, and x^3+1. Ans. $x^{10}-x^6-x^4+1$.
7. $3x^2-11x+6$, $2x^2-7x+3$, and $6x^2-7x+2$. (See Art. 113.) Ans. $6x^3-25x^2+23x-6$.

III. ALGEBRAIC FRACTIONS.

DEFINITIONS.

114. Algebraic Fractions are represented in the same manner as common fractions in arithmetic.

The quantity below the line is called the *denominator*, because it *denominates*, or shows the number of parts into which the unit is divided; the quantity above the line is called the *numerator*, because it *numbers*, or shows how many parts are taken.

Thus, in the fraction $\frac{a-b}{c+d}$, a unit is supposed to be divided into $c+d$ equal parts, and $a-b$ of those parts are taken.

115. The terms *proper*, *improper*, *simple*, *compound*, and *complex*, have the same meaning when applied to algebraic fractions, as to common numerical fractions.

116. An **Entire Algebraic Quantity** is one not expressed under the form of a fraction.

117. A **Mixed Quantity** is one composed of an entire quantity and a fraction.

118. Proposition.—*The value of a fraction is not altered, when both terms are multiplied or divided by the same quantity.*

Let $\frac{A}{B}=Q$. Then, will $\frac{mA}{mB}=Q$. For, since the numerator of a fraction may always be considered a dividend, and the denominator a divisor, if we multiply the numerator or dividend by any quantity, as m, the quotient will be increased m times; if we multiply the denominator or divisor by m, the quotient will be diminished as much, or it will be divided by m. Therefore, the value of the fraction is not changed.

Or, the Proposition may be proved thus:

$$\frac{mA}{mB}=\text{(Art. 81)},\ \frac{m^1m^{-1}A}{B}=\frac{m^0A}{B}=\text{(Art. 82)},\ \frac{A}{B}.$$

A similar method of reasoning may be applied to the division of the terms of a fraction.

Case I.—To reduce a Fraction to its Lowest Terms.

119. From Art. 118, we have the following

Rule.—*Divide both terms of the fraction by any quantity that will exactly divide them, and continue this process as long as possible.*

Or, *Divide both terms by their greatest common divisor.*

Or, *Resolve both terms into their prime factors, and then cancel those factors which are common.*

In algebraic fractions, the last is generally the best method.

1. Reduce $\frac{10acx^2}{15bcx^3}$ to its lowest terms.

$$\frac{10acx^2}{15bcx^3}=\frac{2acx^2}{3bcx^3}=\frac{2ax^2}{3bx^3}=\frac{2a}{3bx}, \text{ Ans.}$$

Or, dividing by $5cx^2$, $\frac{10acx^2}{15bcx^3}=\frac{2a}{3bx}$.

Or, $\frac{10acx^2}{15bcx^3}=\frac{2a\times5cx^2}{3bx\times5cx^2}=\frac{2a}{3bx}$.

2. $\frac{a^2x}{3a^3bx}$. Ans. $\frac{1}{3ab}$.

3. $\frac{ax+x^2}{3bx-cx}$. Ans. $\frac{a+x}{3b-c}$.

4. $\frac{3a^2+3ab}{3a^2-3ab}$. Ans. $\frac{a+b}{a-b}$.

5. $\frac{1-x}{1-x^2}$. Ans. $\frac{1}{1+x}$.

6. $\frac{a^n}{a^{n+1}}$. Ans. $\frac{1}{a}$.

7. $\frac{mnp-m^2p}{m^2p+mp^2}$. Ans. $\frac{n-m}{m+p}$.

8. $\frac{2ax-4ax^2}{6ax}$. Ans. $\frac{1-2x}{3}$.

9. $\frac{5a^2+5ax}{a^2-x^2}$. Ans. $\frac{5a}{a-x}$.

10. $\frac{x^2+2x-3}{x^2+5x+6}$. Ans. $\frac{x-1}{x+2}$.

11. $\frac{x^3-4x^2+5}{x^3+1}$ A. $\frac{x^2-5x+5}{x^2-x+1}$.

12. $\frac{15x^3+35x^2+3x+7}{27x^4+63x^3-12x^2-28x}$. . . . . Ans. $\frac{5x^2+1}{9x^3-4x}$.

The following examples are to be solved by factoring, but the process requires care and practice.

13. Reduce $\frac{x^2+(a+c)x+ac}{x^2+(b+c)x+bc}$ to its lowest terms.

$$x^2+(a+c)x+ac=x^2+ax+cx+ac$$
$$=x(x+a)+c(x+a)=(x+c)(x+a).$$

Also, $x^2+(b+c)x+bc=(x+c)(x+b)$;

$\therefore$ the fraction becomes $\frac{(x+c)(x+a)}{(x+c)(x+b)}=\frac{x+a}{x+b}$, Ans.

14. $\frac{ac+by+ay+bc}{af+2bx+2ax+bf}$. . . . . Ans. $\frac{c+y}{f+2x}$.

15. $\frac{x^8+x^6y^2+x^2y+y^3}{x^4-y^4}$. . . . . Ans. $\frac{x^6+y}{x^2-y^2}$.

16. $\frac{a^3+(a+b)ax+bx^2}{a^4-b^2x^2}$. . . . . Ans. $\frac{a+x}{a^2-bx}$.

17. $\frac{ax^m-bx^{m+1}}{a^2bx-b^3x^3}$. . . . . Ans. $\frac{x^{m-1}}{b(a+bx)}$.

120. Exercises in Division, in which the quotient is a fraction, and capable of being reduced:

1. Divide $2a^3x^2$ by $5a^2x^2b$. . . . . Ans. $\frac{2a}{5b}$.

2. $ax+x^2$ by $3bx-cx$. . . . . . . Ans. $\frac{a+x}{3b-c}$.

3. a^3-b^3 by a^2-b^2. . . . . . Ans. $\frac{a^2+ab+b^2}{a+b}$.

4. a^3-b^3 by $(a-b)^2$. . . . . . Ans. $\frac{a^2+ab+b^2}{a-b}$.

Case II.—To reduce a Fraction to an Entire or Mixed Quantity.

121. Since the numerator of the fraction may be regarded as a dividend, and the denominator as the divisor, this is merely a case of division. Hence,

Rule.—*Divide the numerator by the denominator, for the entire part. If there be a remainder, place it over the denominator, for the fractional part, and reduce it to its lowest terms.*

1. Reduce $\frac{a^3+a^2-ax^2}{a^2-ax}$ to an entire or mixed quantity.

$$\frac{a^3+a^2-ax^2}{a^2-ax}=a+x+\frac{a^2}{a^2-ax}=a+x+\frac{a}{a-x}, \text{ Ans.}$$

Reduce the following to entire or mixed quantities:

2. $\frac{ax-x^2}{a}$. . . . . Ans. $x-\frac{x^2}{a}$.

3. $\frac{a^2+2b^2}{a-b}$. . . . . Ans. $a+b+\frac{3b^2}{a-b}$.

4. $\frac{1+2x}{1-3x}$. . . . . Ans. $1+5x+\frac{15x^2}{1-3x}$.

5. $\frac{x^3+bx^2}{x^2-bx}$. . . . . Ans. $x+\frac{2bx}{x-b}$.

6. $\frac{x^2z^2-z^2+xz-z-x+1}{x^2-1}$. . . . . . . Ans. $z^2+\frac{z-1}{x+1}$.

Case III.—To reduce a Mixed Quantity to the form of a Fraction.

122. This is, obviously, the reverse of Case II. Hence, we have the following

Rule.—1. *Multiply the entire part by the denominator of the fraction.*

2. *Add the numerator to the product, if the sign of the fraction be plus, or subtract it, if the sign be minus.*

3. *Place the result over the denominator.*

Before applying this rule, it is necessary to consider

123. The Signs of Fractions.—Each of the several terms of the numerator and denominator of a fraction is preceded by the sign plus or minus, expressed or understood; and the fraction, taken as a whole, is also preceded by the sign plus or minus, expressed or understood.

Thus, in the fraction $-\frac{a^2-b^2}{x+y}$, the sign of a^2 is plus; of b^2, minus; while the sign of each term of the denominator is plus; but the sign of the fraction, taken as a whole, is minus.

124. It is often convenient to change the signs of the numerator or denominator of a fraction, or both.

By the rule for the signs, in Division (Art. 69), we have,

$\frac{+ab}{+a}=+b$; or, changing the signs of *both* terms, $\frac{-ab}{-a}=+b$.

If we change the sign of the *numerator*, we have $\frac{-ab}{+a}=-b$.

If we change the sign of the *denominator*, we have $\frac{+ab}{-a}=-b$. Hence,

1. *The signs of both terms of a fraction may be changed, without altering its value or changing its sign, as a whole.*

2. *If the sign of either term be changed, the sign of the fraction will be changed.* Hence, also,

3. *The signs of either term of a fraction may be changed, without altering its value, if the sign of the fraction be changed at the same time.*

Thus, $\frac{a^2-x^2}{a-x}=-\frac{-a^2+x^2}{a-x}=-\frac{a^2-x^2}{-a+x}=-(-a-x)=a+x$.

And, $a-\frac{a^2-x^2}{a-x}=a+\frac{-a^2+x^2}{a-x}=a+\frac{a^2-x^2}{-a+x}=-x$.

Applying the above principles, the *sign of the fraction* may be made plus, in all cases, if desired.

Reduce the following quantities to a fractional form:

1. $2+\frac{3}{5}$ and $2-\frac{3}{5}$. . . . . Ans. $\frac{13}{5}$ and $\frac{7}{5}$.

2. $a+x+\frac{a^2-ax}{x}$. . . . . Ans. $\frac{a^2+x^2}{x}$.

3. $a^2-ax+x^2-\frac{2x^3}{a+x}$. . . . . Ans. $\frac{a^3-x^3}{a+x}$.

4. $2a-x+\frac{(a-x)^2}{x}$. . . . . Ans. $\frac{a^2}{x}$.

5. $a-\frac{a^2}{a+b}$. . . . . Ans. $\frac{ab}{a+b}$.

6. $a-x-\frac{a^2+x^2}{a+x}$. . . . . Ans. $-\frac{2x^2}{a+x}$.

7. $1-\frac{(x-y)^2}{x^2+y^2}$. . . . . Ans. $\frac{2xy}{x^2+y^2}$.

Case IV.—To reduce Fractions of Different Denominators to Equivalent Fractions having a Common Denominator.

125.—1. Let it be required to reduce $\frac{a}{m}$, $\frac{b}{n}$, and $\frac{c}{r}$, to a common denominator.

If we multiply both terms of the first fraction by nr, of the second by mr, and of the third by mn, we have

$$\frac{anr}{mnr},\ \frac{bmr}{mnr},\ \text{and}\ \frac{cmn}{mnr}.$$

As the terms of each fraction have thus been multiplied by the same quantity, the value of the fractions has not been changed. (Art. 118.) Hence,

TO REDUCE FRACTIONS TO A COMMON DENOMINATOR,

Rule.—*Multiply both terms of each fraction by the product of all the denominators, except its own.* Or,

1. *Multiply each numerator by the product of all the denominators except its own, for the new numerators.*

2. *Multiply all the denominators together for the common denominator.*

Reduce the fractions in each of the following to a common denominator:

2. $\frac{1}{x}$, $\frac{2}{y}$, and $\frac{3}{z}$. . . . . Ans. $\frac{yz}{xyz}$, $\frac{2xz}{xyz}$, $\frac{3xy}{xyz}$.

3. $\frac{a}{b}$ and $\frac{b}{a}$. . . . . Ans. $\frac{a^2}{ab}$ and $\frac{b^2}{ab}$.

4. $\frac{x}{x-a}$ and $\frac{a}{x+a}$. . . . . . Ans. $\frac{x^2+ax}{x^2-a^2}$ and $\frac{ax-a^2}{x^2-a^2}$.

126. It frequently happens, that the denominators of the fractions to be reduced contain a common factor. In such cases the preceding rule does not give the *least* common denominator.

1. Let it be required to reduce $\frac{a}{m}$, $\frac{b}{mn}$, and $\frac{c}{nr}$, to their least common denominator.

Since the denominators of these fractions contain only three prime factors, m, n, and r, it is evident that the *least* common denominator will contain these three factors, and no others; that is, it will be mnr, the L.C.M. of m, mn, and nr.

It now remains to reduce each fraction, without altering its value, to another whose denominator shall be mnr.

To effect this, we must multiply both terms, of the first fraction by nr, of the second by r, and of the third by m. But these multipliers will evidently be obtained by dividing mnr by m, mn, and nr; that is, by dividing the L.C.M. of the given denominators by the several denominators. Hence,

TO REDUCE FRACTIONS OF DIFFERENT DENOMINATORS TO EQUIVALENT FRACTIONS HAVING THE LEAST COMMON DENOMINATOR,

Rule.—1. *Find the L.C.M. of all the denominators; this will be the common denominator.*

2. *Divide the L.C.M. by the first of the given denominators, and multiply the quotient by the first of the given numerators; the product will be the first of the required numerators.*

3. *Proceed thus to find each of the other numerators.*

Reduce the fractions, in each of the following, to equivalent fractions having the least common denominator:

2. $\frac{a}{6xy}, \frac{b}{3x}, \frac{c}{2y}$. . . . . Ans. $\frac{a}{6xy}, \frac{2by}{6xy}, \frac{3cx}{6xy}$.

3. $\frac{x}{a+b}, \frac{y}{a-b}, \frac{z}{a^2-b^2}$. Ans. $\frac{x(a-b)}{a^2-b^2}, \frac{y(a+b)}{a^2-b^2}, \frac{z}{a^2-b^2}$.

4. $\frac{m-n}{m+n}$, $\frac{m+n}{m-n}$, $\frac{m^2n^2}{m^2-n^2}$. Ans. $\frac{(m-n)^2}{m^2-n^2}$, $\frac{(m+n)^2}{m^2-n^2}$, $\frac{m^2n^2}{m^2-n^2}$.

Other exercises will be found in Addition of Fractions.

NOTE.—The two following Articles may be of frequent use.

127. To reduce an entire quantity to the form of a fraction having a given denominator,

Rule.—*Multiply the entire quantity by the given denominator, and write the product over it.*

1. Reduce x to a fraction whose denominator is a.

Ans. $\frac{ax}{a}$.

2. Reduce $2az$ to a fraction whose denominator is z^2.

Ans. $\frac{2az^3}{z^2}$.

3. Reduce $x+y$ to a fraction whose denominator is $x-y$.

Ans. $\frac{x^2-y^2}{x-y}$.

128. To convert a fraction to an equivalent one having a given denominator,

Rule.—*Divide the given denominator by the denominator of the given fraction, and multiply both terms by the quotient.*

1. Convert $\frac{3}{7}$ to an equivalent fraction, having 49 for its denominator. Ans. $\frac{21}{49}$.

2. Convert $\frac{a}{3}$ and $\frac{5}{c}$ to equivalent fractions having the denominator $9c^2$. Ans. $\frac{3ac^2}{9c^2}$ and $\frac{45c}{9c^2}$.

3. Convert $\frac{a+b}{a-b}$ and $\frac{a-b}{a+b}$ to equivalent fractions having the denominator a^2-b^2. Ans. $\frac{(a+b)^2}{a^2-b^2}$, $\frac{(a-b)^2}{a^2-b^2}$.

Case V.—ADDITION AND SUBTRACTION OF FRACTIONS.

129.—1. Required to find the value of $\frac{a}{d}$, $\frac{b}{d}$, and $\frac{c}{d}$.

Since in each of these fractions the unit is supposed to be divided into d parts, it is evident that their sum will be expressed by the fraction $\frac{a+b+c}{d}$. Hence,

Rule for the Addition of Fractions.—1. *Reduce the fractions, if necessary, to a common denominator.*

2. *Add the numerators, and write their sum over the common denominator.*

130.—2. Let it be required to subtract $\frac{b}{d}$ from $\frac{a}{d}$.

The unit being, in each case, divided into the same parts, the difference will evidently be expressed by $\frac{a-b}{d}$. Hence,

Rule for the Subtraction of Fractions.—1. *Reduce the fractions, if necessary, to a common denominator.*

2. *Subtract the numerator of the subtrahend from the numerator of the minuend, and write the remainder over the common denominator.*

EXAMPLES IN ADDITION OF FRACTIONS.

1. Add $\frac{a}{b}$ and $\frac{3a}{4b}$ together. . . . . Ans. $\frac{7a}{4b}$.

2. Add $\frac{a}{b}$ and $\frac{b}{a}$ together. . . . . . Ans. $\frac{a^2+b^2}{ab}$.

3. Add $\frac{1}{1+x}$ and $\frac{1}{1-x}$ together. . . . Ans. $\frac{2}{1-x^2}$.

4. Find the value of $\frac{c}{x}+\frac{b}{x^3}+\frac{a}{x^5}$. . Ans. $\frac{a+bx^2+cx^4}{x^5}$.

5. Find the value of $\frac{b}{d}+\frac{ad-bc}{d(c+dx)}$. . . Ans. $\frac{a+bx}{c+dx}$.

6. Find the value of $\frac{p}{ab}+\frac{q}{ac}+\frac{r}{bc}$. Ans. $\frac{pc+qb+ra}{abc}$.

7. Of $\frac{x}{x+y}+\frac{y}{x-y}$. . . . . Ans. $\frac{x^2+y^2}{x^2-y^2}$.

8. Of $\frac{1}{4(1+x)}+\frac{1}{4(1-x)}+\frac{1}{2(1+x^2)}$. Ans. $\frac{1}{1-x^4}$.

9. Of $\frac{p-q}{pq}+\frac{r-p}{pr}+\frac{q-r}{qr}$. . . . . Ans. 0.

10. Of $\frac{1}{4a^3(a+x)}+\frac{1}{4a^3(a-x)}+\frac{1}{2a^2(a^2+x^2)}$.

Ans. $\frac{1}{a^4-x^4}$.

11. Of $\frac{1}{a(a-b)(a-c)}+\frac{1}{b(b-a)(b-c)}+\frac{1}{c(c-a)(c-b)}$.

Ans. $\frac{1}{abc}$.

EXAMPLES IN SUBTRACTION OF FRACTIONS.

Subtract the second fraction, in the following, from the first:

1. $\frac{5x}{7a}$ and $\frac{3y}{7}$. . . . . Ans. $\frac{5x-3ay}{7a}$.

2. $\frac{1}{a-b}$ and $\frac{1}{a+b}$. . . . . Ans. $\frac{2b}{a^2-b^2}$.

3. $\frac{p+q}{p-q}$ and $\frac{p-q}{p+q}$. . . . . Ans. $\frac{4pq}{p^2-q^2}$.

4. $\frac{n-1}{n}$ and $\frac{n}{n-1}$. . . . . Ans. $\frac{1-2n}{n^2-n}$.

5. $\frac{1}{1-x}$ and $\frac{2}{1-x^2}$ . . . . . . Ans. $\frac{x-1}{1-x^2}=-\frac{1}{1+x}$.

6. $\frac{1}{(x+1)(x+2)}$ and $\frac{1}{(x+1)(x+2)(x+3)}$.

Ans. $\frac{1}{(x+1)(x+3)}$.

7. $\frac{a}{c}$ and $\frac{(ad-bc)x}{c(c+dx)}$. . . . . . . Ans. $\frac{a+bx}{c+dx}$.

8. $\frac{1}{2}\cdot\frac{3m+2n}{3m-2n}$ and $\frac{1}{2}\cdot\frac{3m-2n}{3m+2n}$. . . . Ans. $\frac{12mn}{9m^2-4n^2}$.

9. $\frac{a+c}{(a-b)(x-a)}$ and $\frac{b+c}{(a-b)(x-b)}$. A. $\frac{x+c}{(x-a)(x-b)}$.

Find the value

10. Of $\frac{4m-3n}{3(1-n)}-\frac{m+3n}{3(1-n)}+\frac{2n}{1-n}$. . . . Ans. $\frac{m}{1-n}$.

11. Of $\frac{a-b}{ab}-\frac{a-c}{ac}+\frac{b-c}{bc}$. . . . . Ans. 0.

12. Of $\frac{x+y}{y}-\frac{x}{x+y}-\frac{x^3-x^2y}{x^2y-y^3}$. . . . . . . Ans. 1.

13. Of $\frac{1}{x-1}-\frac{1}{2(x+1)}-\frac{x+3}{2(x^2+1)}$. . . Ans. $\frac{x+3}{x^4-1}$.

Case VI.—MULTIPLICATION OF FRACTIONS.

131.—1. Required to find the product of $\frac{a}{b}$ by $\frac{c}{d}$.

Here, as in arithmetic, we take the part of $\frac{a}{b}$, which is expressed by $\frac{1}{d}$, and then multiply by c. Thus, the $\frac{1}{d}$ part of $\frac{a}{b}$ is $\frac{a}{bd}$, and c times $\frac{a}{bd}$ is $\frac{ac}{bd}$ (Art. 118). Hence, $\frac{a}{b}\times\frac{c}{d}=\frac{ac}{bd}$.

Or thus, $\frac{a}{b}$ and $\frac{c}{d}=ab^{-1}$ and cd^{-1} (Art. 81). Multiplying, we have $ab^{-1}cd^{-1}=\frac{ac}{bd}$. Hence,

Rule.—*Multiply the numerators together for a new numerator, and the denominators together for a new denominator*

REMARKS.—1st. To multiply a fraction by an integral quantity, reduce the latter to the form of a fraction, by writing unity beneath it; or, multiply the numerator by the integer.

2d. If either of the factors is a mixed quantity, reduce it to an improper fraction.

3d. When the numerators and denominators have common factors, let such factors be first separated, and then canceled.

Thus, $\frac{2a^2}{a^2-b^2}\times\frac{(a+b)^2}{4a^2b}=\frac{2a^2\times(a+b)(a+b)}{(a+b)(a-b)4a^2b}=\frac{a+b}{2b(a-b)}$.

Find the products of the fractions in the following:

1. $\frac{3x}{4}$ by $\frac{4x}{3y}$ and $\frac{8a^2b}{c}$ by $\frac{c^2d}{8a^3}$. . . . Ans. $\frac{x^2}{y}$ and $\frac{bcd}{a}$.

2. $a-\frac{x^2}{a}$, $\frac{a}{x}+\frac{x}{a}$. . . . . Ans. $\frac{a^4-x^4}{a^2x}$.

3. $1-\frac{x-y}{x+y}$ and $2+\frac{2y}{x-y}$. . . . . Ans. $\frac{4xy}{x^2-y^2}$.

4. $\frac{1+a+a^2}{1-b+b^2}$ and $\frac{1-a}{1+b}$. . . . . Ans. $\frac{1-a^3}{1+b^3}$.

5. $\frac{x^2+3x+2}{x^2+2x+1}$ and $\frac{x^2+5x+4}{x^2+7x+12}$. . . . . . Ans. $\frac{x+2}{x+3}$.

6. $\frac{4ax}{3by}$, $\frac{a^2-x^2}{c^2-x^2}$, $\frac{bc+bx}{a^2-ax}$. . . . . Ans. $\frac{4x(a+x)}{3y(c-x)}$.

7. $\frac{a^2-b^2}{x+y}$, $\frac{x^2-y^2}{a-b}$, $\frac{a^2}{(x-y)^2}$. . . . . Ans. $\frac{a^2(a+b)}{x-y}$.

8. $x+1+\frac{1}{x}$ by $x-1+\frac{1}{x}$. . . . . . Ans. $x^2+1+\frac{1}{x^2}$.

9. $\frac{4a}{3x}+\frac{3x}{2b}$ by $\frac{2b}{3x}+\frac{3x}{4a}$. . . . . . Ans. $\frac{8ab}{9x^2}+2+\frac{9x^2}{8ab}$.

10. $\frac{pr+(pq+qr)x+q^2x^2}{p-qx}$ by $\frac{ps+(pt-qs)x-qtx^2}{p+qx}$.

Ans. $rs+(rt+qs)x+qtx^2$.

Find the value

11. Of $\left(\frac{a}{b}+\frac{b}{a}\right)\left(\frac{c}{d}+\frac{d}{c}\right)-\left(\frac{a}{b}-\frac{b}{a}\right)\left(\frac{c}{d}-\frac{d}{c}\right)$.

Ans. $\frac{2bc}{ad}+\frac{2ad}{bc}$.

Case VII.—DIVISION OF FRACTIONS.

132.—1. Required to find the quotient of $\frac{a}{b}$ by $\frac{c}{d}$.

Here, as in arithmetic, the quotient of $\frac{a}{b}$ by $\frac{1}{d}$ is $\frac{ad}{b}$, and the quotient of $\frac{a}{b}$ by c times $\frac{1}{d}$, or $\frac{c}{d}$, is $\frac{ad}{b}$ divided by c, or $\frac{ad}{bc}$.

Or thus, $\frac{a}{b}$ and $\frac{c}{d}$= (Art. 81) ab^{-1} and cd^{-1}. Dividing, we have $\frac{ab^{-1}}{cd^{-1}}=\frac{ad}{bc}$. Hence,

Rule.—*Invert the divisor, and proceed as in multiplication of fractions.*

Remark.—To divide a fraction by an integral quantity, reduce the latter to the form of a fraction, by writing unity beneath it; or, multiply the denominator by the integer.

Remarks 2 and 3, Art. 131, apply equally well to division of fractions.

Required, in their simplest forms, the quotients

1. Of $\frac{ab^2c^3}{x^2y}\div\frac{a^3b^2c}{xy^2}$. . . . . Ans. $\frac{c^2y}{a^2x}$.

2. Of $\frac{a+b}{a+c}\div\frac{a-c}{a-b}$. . . . . Ans. $\frac{a^2-b^2}{a^2-c^2}$.

3. Of $\frac{x^3-a^2x}{a^2}\div\frac{ax-a^2}{x}$. . . . . Ans. $\frac{x^3+ax^2}{a^3}$.

4. Of $\left(1+\frac{1}{a}\right)\div\left(1-\frac{1}{a^2}\right)$. . . . . . Ans. $\frac{a}{a-1}$.

5. Of $\frac{x^3+y^3}{x^2-y^2}\div\frac{x^2-xy+y^2}{x-y}$. . . . . Ans. 1.

6. Of $\frac{a^4-x^4}{a^2-2ax+x^2}\div\frac{a^2x+x^3}{a^3-x^3}$. . Ans. $\frac{a+x}{x}(a^2+ax+x^2)$.

7. Of $\left(\frac{1}{1+x}+\frac{x}{1-x}\right)\div\left(\frac{1}{1-x}-\frac{x}{1+x}\right)$. . Ans. 1.

8. Of $\frac{3x}{2x-2}\div\frac{2x}{x-1}$. . . . . Ans. $\frac{3}{4}$.

9. Of $\left(x+\frac{2x}{x-3}\right)\div\left(x-\frac{2x}{x-3}\right)$. . . . Ans. $\frac{x-1}{x-5}$.

10. Of $\left(x^4-\frac{1}{x^4}\right)\div\left(x-\frac{1}{x}\right)$. . . . Ans. $x^3+\frac{1}{x^3}+x+\frac{1}{x}$.

To reduce a Complex Fraction to a Simple one.

133. This is merely a case of division, in which the dividend and divisor are either fractions or mixed quantities.

Thus, $\dfrac{a+\frac{b}{c}}{m-\frac{n}{r}}$ is the same as to divide $a+\frac{b}{c}$ by $m-\frac{n}{r}$.

$$\left(a+\frac{b}{c}\right)\div\left(m-\frac{n}{r}\right)=\frac{ac+b}{c}\div\frac{mr-n}{r}=\frac{ac+b}{c}\times\frac{r}{mr-n}$$
$$=\frac{acr+br}{cmr-cn}.$$

Or, the following method, obviously true, will generally be found more convenient.

Multiply both terms of the complex fraction by the product of the denominators, or by their L.C.M.

Thus, in the above, multiplying by cr, we have, at once, $\frac{acr+br}{cmr-cn}$.

Solve the following examples by both methods:

1. $\dfrac{\frac{3x}{2x-2}}{\frac{2x}{x-1}}$. . . . Ans. $\frac{3}{4}$.

2. $\dfrac{\frac{a}{b}+\frac{c}{d}}{\frac{e}{f}-\frac{g}{h}}$. Ans. $\frac{fh(ad+bc)}{bd(eh-fg)}$.

3. $\dfrac{\frac{a+1}{a-1}+\frac{a-1}{a+1}}{\frac{a+1}{a-1}-\frac{a-1}{a+1}}$. Ans. $\frac{a^2+1}{2a}$.

4. $\dfrac{a+b+\frac{b^2}{a}}{a+b+\frac{a^2}{b}}$. . . . Ans. $\frac{b}{a}$.

Resolution of Fractions into Series.

134. An **Infinite Series** consists of an unlimited number of terms which observe the same law.

The **Law of a Series** is a relation existing between its terms, such as that when some of them are known the others may be found.

Thus, in the infinite series $1-\frac{1}{x}+\frac{1}{x^2}-\frac{1}{x^3}+$, etc., any term may be found by multiplying the preceding term by $-\frac{1}{x}$.

Any proper algebraic fraction, whose denominator is a polynomial, may, by division, be resolved into an infinite series.

1. Convert the fraction $\frac{1-x}{1+x}$ into an infinite series.

$$\begin{array}{l|l}
 & 1-x \\
\hline
1+x & 1-2x+2x^2-2x^3+\text{, etc.} \\
\hline
 & -2x \\
 & -2x-2x^2 \\
\hline
 & \quad +2x^2 \\
 & \quad +2x^2+2x^3 \\
\hline
 & \qquad -2x^3
\end{array}$$

It is evident that the *law* of this series is, that each term, after the second, is equal to the preceding term, multiplied by $-x$.

Resolve the following fractions into infinite series:

2. $\frac{1}{1+r^2}=1-r^2+r^4-r^6+r^8-$, etc., to infinity.

3. $\frac{1}{1-r+r^2}=1+r-r^3-r^4+r^6+r^7-r^9-r^{10}+$, etc.

4. $\frac{a}{a+b}=1-\frac{b}{a}+\frac{b^2}{a^2}-\frac{b^3}{a^3}+$, etc.

135. Miscellaneous Propositions in Fractions.—The answer to some general question, that is, the solution to a *literal* equation (cf. Arts. 162–165), may happen to be a fraction: *e. g.*, we may have $x=\frac{a}{b}$. When the two terms of the fraction are finite numbers, the fraction, being the ratio of two finite numbers, has a determinate value. But

the values of the numerator and denominator may be changed, by reason of some suppositions as to the values of the known numbers involved in the question, thus giving rise to anomalous results requiring explanation.

136.—1. Thus, suppose $x=\frac{a}{b}$. If, while the denominator remains constant, the numerator changes, the value of the fraction varies *directly* with the numerator: if a decreases, the fraction decreases; if a becomes 0, the fraction likewise becomes 0, that is, $\frac{0}{b}=0$. Also, if a increases, b being constant, the fraction increases; if a becomes ∞, the fraction becomes ∞; that is, $\frac{\infty}{b}=\infty$.

2. If the denominator changes while the numerator remains constant, the value of the fraction varies *inversely;* that is, if b decreases, the value of the fraction *increases*, and, *vice versa*, if b increases, the value of the fraction decreases; if b becomes 0, the fraction becomes ∞, or $\frac{a}{0}=\infty$; if, on the other hand, b becomes ∞, the fraction becomes 0, or $\frac{a}{\infty}=0$.

137. If the numerator and denominator are both rendered zero simultaneously, the solution assumes the form $x=\frac{0}{0}$. In this case, the unknown number x is said to be *indeterminate*, inasmuch as it may evidently, at this stage of the investigation, have any value whatever; since the only condition imposed upon it is, that it shall give, when multiplied by zero, a product equal to zero. Hence, the form $\frac{0}{0}$ has been called the *symbol of indetermination.* Nevertheless, it may, and indeed generally does, happen that the indetermination is only *apparent*, being due to the presence, in numerator and denominator, of a common factor which the particular hypothesis reduces to zero, and which

if suppressed *before* making the hypothesis, will leave the result in a determinate form. Thus, suppose some equation has given us $x=\frac{a^2-b^2}{a-b}$: if we make $b=a$, we have $x=\frac{0}{0}$: if, however, we cancel the common factor $a-b$, and *then* make $b=a$, we have $x=2a$. So, if $x=\frac{a^2-1}{a^2+a-2}$: for $a=1$, we have $x=\frac{0}{0}$: but cancelling the common factor $a-1$ before making $a=1$, gives $x=\frac{2}{3}$.

These considerations show that it is not safe to assume the symbol $\frac{0}{0}$ as indicating *absolute* indetermination, until we have ascertained whether the result has not been caused, as in the examples cited, by the presence of a common factor which becomes zero under the particular supposition imposed. Finally, it should be remembered that all of these symbols, discussed in this and the preceding article, as well as some others of the same character, omitted as unsuited to an elementary work, are to be interpreted as mere *abbreviations;* otherwise, they are *without meaning.*

138. Theorem.—*If the same quantity be added to both terms of a proper fraction, the new fraction resulting will be greater than the first; but if the same quantity be added to both terms of an improper fraction, the new fraction resulting will be less than the first.*

Let m represent the quantity to be added to each term of any fraction, as $\frac{a}{b}$; then, the resulting fraction is $\frac{a+m}{b+m}$.

Reducing $\frac{a}{b}$ and $\frac{a+m}{b+m}$ to a common denominator, we have

$$\frac{ab+am}{b^2+bm} \text{ and } \frac{ab+bm}{b^2+bm}.$$

Since the denominators are the same, that fraction which has the greater numerator is the greater. Now, if $\frac{a}{b}$ is a proper fraction, or if a is less than b, the second fraction is obviously greater; but if it is improper, and a greater than b, the second is less than the first; which proves the theorem.

139. Theorem.—*If the same quantity be subtracted from both terms of a proper fraction, the new fraction resulting will be less than the first; but if the same quantity be subtracted from both terms of an improper fraction, the new fraction resulting will be greater than the first.*

Let m represent the quantity to be subtracted from each term of any fraction, as $\frac{a}{b}$; then, the resulting fraction is $\frac{a-m}{b-m}$.

Reducing these fractions to a common denominator, we have

$$\frac{ab-am}{b^2-bm} \text{ and } \frac{ab-bm}{b^2-bm}.$$

Reasoning as in the preceding theorem, when the original fraction is proper, the second fraction is evidently less than the first; when improper, it is greater.

MISCELLANEOUS EXERCISES.

1. Prove that $\frac{x}{x-3}-\frac{x-3}{x}+\frac{x}{x+3}-\frac{x+3}{x}=\frac{18}{x^2-9}$.

2. $$\frac{a^2+a+1}{(a-b)(a-c)}+\frac{b^2+b+1}{(b-a)(b-c)}+\frac{c^2+c+1}{(c-a)(c-b)}=1.$$

3. Find the value of $\left(x+\frac{2x}{x-3}\right)\div\left(x-\frac{2x}{x-3}\right)$, when $x=5\frac{1}{2}$. Ans. 9.

4. Of $\frac{x+2a}{x-2a}+\frac{x+2b}{x-2b}$, when $x=\frac{4ab}{a+b}$. Ans. 2.

5. Prove that the sum or difference of any two quantities, divided by their product, is equal to the sum or difference of their reciprocals.

6. If $\frac{a^2+h^2}{(a-b)(a-c)}+\frac{b^2+h^2}{(b-a)(b-c)}+\frac{c^2+h^2}{(c-a)(c-b)}=1$, prove that when the terms are multiplied respectively by $b+c$, $a+c$, and $a+b$, the sum $=0$; and that when multiplied respectively by bc, ac, and ab, it is $=h^2$.

IV. SIMPLE EQUATIONS.

DEFINITIONS AND ELEMENTARY PRINCIPLES.

140. An **Equation** is an algebraic expression, stating the equality between two quantities. Thus,

$$x-5=3$$

is an equation, stating that if 5 be subtracted from x, the remainder will be 3.

141. Every equation is composed of two parts, separated from each other by the sign of equality.

The **First Member** of an equation is the quantity on the left of the sign of equality.

The **Second Member** is the quantity on the right of the sign of equality.

Each member of an equation is composed of one or more terms.

142. There are generally two classes of quantities in an equation, the *known* and the *unknown*.

The **Known Quantities** are represented either by numbers or the first letters of the alphabet; as, a, b, c, etc.

The **Unknown Quantities** are represented by the last letters of the alphabet; as, x, y, z, etc.

143. Equations are divided into degrees, called *first*, *second*, *third*, and so on.

The **Degree** of an equation depends on the highest power of the unknown quantity which it contains.

A **Simple Equation,** or *an equation of the first degree*, is one that contains no power of the unknown quantity higher than the *first*.

A **Quadratic Equation,** or *an equation of the second degree*, is one in which the highest power of the unknown quantity is a *square.*

Similarly, we have equations of the *third* degree, *fourth* degree, and so on. Those of the *third* degree are generally called *cubic* equations; and those of the *fourth* degree, *biquadratic* equations. Thus,

$ax-b=c$, is an equation of the 1st degree.						
$x^2+2px=q$, "	"	"	2d	"	or quadratic equation.	
$x^3-px=q$, "	"	"	3d	"	or cubic	"
$x^4+ax^3+px=q$,	"	"	4th	"	or biquadratic	"
$x^n+ax^{n-1}+bx^{n-2}=c$,		"	nth degree.			

When any equation contains more than one unknown quantity, its degree is equal to the greatest sum of the exponents of the unknown quantity, in any of its terms.

Thus, $xy+ax-by=c$, is an equation of the 2d degree.
$x^2y+x^2-cx=a$, is an equation of the 3d degree.

144. A **Complete Equation** of any degree is one that contains all the powers of the unknown quantity, from 0 up to the given degree.

An **Incomplete Equation** is an equation in which one or more terms are wanting.

Thus, $x^2+px+q=0$, is a complete equation of the second degree, the term q being equivalent to qx^0, since $x^0=1$. Art. 82.

$x^3+px^2+qx+r=0$, is a complete equation of the third degree.

$ax^2=q$, is an incomplete equation of the second degree.

$x^3+px=q$, is an incomplete equation of the third degree.

145. An **Identical Equation** is one in which the two members are identical; or, one in which one of the members is the result of the operations indicated in the other.

Thus, $ax-b=ax-b$,
$8x-3x=5x$,
$(x+3)(x-3)=x^2-9$, are identical equations.

Equations are also distinguished as *numerical* and *literal.*

A **Numerical Equation** is one in which all the known quantities are expressed by numbers; as, $2x^2+3x=10x+15$.

A **Literal Equation** is one in which the known quantities are represented by letters, or by letters and numbers; as, $ax+b=cx+d$, and $ax+b=3x+5$.

146. Every equation may be regarded as the statement, in algebraic language, of a particular question.

Thus, $x-5=9$, may be regarded as the statement of the following question: To find a number from which, if 5 be subtracted, the remainder shall be 9.

To Solve an Equation is *to find the value of the unknown quantity.*

An equation is said to be *verified* when the value of the unknown quantity, being substituted for it, the two members are rendered equal to each other.

Thus, in the equation $x-5=9$, if 14, which is the true value of x, be substituted instead of it,

We have, $14-5=9$;

Or, $9=9$.

147. The value of the unknown quantity, in any equation, is called the *root* of that equation.

SIMPLE EQUATIONS CONTAINING ONE UNKNOWN QUANTITY.

148. All the rules employed in the solution of equations are founded on this evident principle:

If we perform the same operation on two equal quantities, the results will be equal.

This principle may be otherwise expressed in the following self-evident propositions, or

AXIOMS.

1. *If, to two equal quantities, the same quantity be added, the sums will be equal.*

2. *If, from two equal quantities, the same quantity be subtracted, the remainders will be equal.*

3. *If two equal quantities be multiplied by the same quantity, the products will be equal.*

4. *If two equal quantities be divided by the same quantity, the quotients will be equal.*

5. *If two equal quantities be raised to the same power, the results will be equal.*

6. *If the same root of two equal quantities be extracted, the results will be equal.*

149. There are two operations of constant use in the solution of equations. These are *Transposition*, and *Clearing an Equation of Fractions.*

TRANSPOSITION.

150. Suppose we have the equation $x-b=c$.

By Axiom 1, Art. 148, we may add any quantity to both members of this equation without destroying the equality. Adding b to both sides,

We have, $x-b+b=c+b$;
Or, $x=c+b$, since $-b+b=0$.

Comparing this result with the original equation, we find that it is the same as if we had removed the term b to the other side of the equation, with its sign changed.

Again, take the equation $x+b=c$.
Subtracting b from both sides, Ax. 2, $x+b-b=c-b$;
Or, $x=c-b$.

Here again we have the same result as if we had transposed b to the other side with its sign changed. The same method may be employed in removing a term from the second member of the equation to the first. Hence,

Rule of Transposition.—*Any quantity may be transposed from one side of an equation to the other, if, at the same time, its sign be changed.*

151. To Clear an Equation of Fractions.—1. Let it be required to clear the following equation of fractions:

$$\frac{x}{ab}-\frac{x}{bc}=d.$$

Since, by Ax. 3, Art. 148, we may multiply both members of this equation by any quantity without destroying the equality, we first multiply by ab, the denominator of the first fraction.

This gives, . . $x-\frac{abx}{bc}=abd.$

Multiplying both members again by bc,

We have, . . $bcx-abx=ab^2cd.$ (1)

Dividing both members by b, Ax. 4, Art. 148,

We have, . $cx-ax=abcd.$ (2)

If, instead of multiplying *successively* by ab and bc, we had, at once, multiplied by $ab\times bc$, or ab^2c, we would have obtained the form (1) by one operation. By multiplying both members by abc, the L.C M. of the denominators, we would have obtained the *reduced* form (2). Of these three methods, the third is the most simple. Hence,

Rule for Clearing an Equation of Fractions.—*Find the L.C.M. of all the denominators, and multiply each term of the equation by it.*

Clear the following equations of fractions:

2. $\frac{x}{3}-\frac{x}{4}=1.$ . . . . Ans. $4x-3x=12.$

3. $\frac{x}{4}+\frac{x}{6}=5.$ . . . . Ans. $3x+2x=60.$

4. $\frac{x}{4}-\frac{x}{8}+\frac{x}{12}=3\frac{1}{2}.$ . . . . . Ans. $6x-3x+2x=84.$

5. $2x-\frac{x-3}{5}=\frac{x-3}{2}$. . Ans. $20x-2x+6=5x-15$.

When a fraction, whose denominator is to be removed, is preceded by a minus sign, the signs of all the terms in the numerator must be changed. See Art. 46, 2d. Thus, in the above example, we have $20x-(2x-6)=5x-15$, or $20x-2x+6=5x-15$.

6. $x-\frac{x-2}{4}=5-\frac{x+2}{6}$. A. $12x-3x+6=60-2x-4$.

7. $\frac{x}{ab}+\frac{ax}{bc}-\frac{bx}{ac}=m$. . . Ans. $cx+a^2x-b^2x=abcm$.

8. $\frac{x-a}{a+b}-\frac{x-a}{a-b}=\frac{2nb}{a^2-b^2}$.

Ans. $ax-a^2-bx+ab-ax+a^2-bx+ab=2nb$.

SOLUTION OF SIMPLE EQUATIONS CONTAINING ONLY ONE UNKNOWN QUANTITY.

152. The unknown quantity in an equation may be combined with the known quantities, either by *addition*, *subtraction*, *multiplication*, or *division;* or by two or more of these different methods.

1. Let it be required to find the value of x, in the equation

$$a+x=b,$$

where the unknown quantity is connected by *addition*.

By *subtracting* a from each side (Art. 148), we have

$$x=b-a.$$

2. Let it be required to find the value of x, in the equation

$$x-a=b,$$

where the unknown quantity is connected by *subtraction*.

By *adding* a to each side (Art. 148), we have

$$x=b+a.$$

3. Let it be required to find the value of x, in the equation

$$ax=b,$$

where the unknown quantity is connected by *multiplication*.

By *dividing* each side by a, we have

$$x=\frac{b}{a}.$$

4. Let it be required to find the value of x, in the equation

$$\frac{x}{a}=b,$$

where the unknown quantity is connected by *division.*

By *multiplying* each side by a, we have

$$x=b\times a=ab.$$

From the solution of these examples, we see that

When the unknown quantity is connected by addition, it is to be separated by subtraction.

When connected by subtraction, it is separated by addition.

When connected by multiplication, it is separated by division.

When connected by division, it is separated by multiplication.

5. Let it be required to find the value of x, in the equation

$$3x-\frac{24-2x}{7}=x+8.$$

Clearing of fractions,	$21x-(24-2x)=7x+56$,
Or,	$21x-24+2x=7x+56$.
Transposing,	$21x+2x-7x=56+24$;
Reducing,	$16x=80$;
Dividing by 16,	$x=\frac{80}{16}=5$.

In this solution there are three steps, viz.: 1st. *Clearing the equation of fractions;* 2d. *Transposition;* and 3d. *Reducing like terms, and dividing by the coëfficient of x.*

Let the value of x be substituted instead of x in the original equation, and, if it is the true value, the two members will be equal to each other. This is called *verification.*

Original equation, $3x-\frac{24-2x}{7}=x+8.$

Substituting 5 for x, $3\times5-\frac{24-2\times5}{7}=5+8;$

Or, $15-2=5+8;$ or, $13=13.$

6. Find the value of x, in the equation

$$x-\frac{x+a}{ab}=d+\frac{x}{bc}.$$

1st *step*, $abcx-cx-ac=abcd+ax$.
2d *step*, $abcx-cx-ax=abcd+ac$.
Factoring, $(abc-c-a)\,x=ac(bd+1)$.
3d *step*, $x=\dfrac{ac(bd+1)}{abc-c-a}$.

153. From the solution of the preceding examples, we derive the following

Rule for the Solution of a Simple Equation.—1. *If necessary, clear the equation of fractions, and perform all the operations indicated.*

2. *Transpose all the terms containing the unknown quantity to one side, and the known quantities to the other.*

3. *Reduce each member to its simplest form.*

4. *Divide both sides by the coëfficient of the unknown quantity.*

Find the value of the unknown quantity in the following:

7. $\dfrac{3x+7}{14}-\dfrac{2x-7}{21}+2\frac{3}{4}=\dfrac{x-4}{4}$.

1st *step*, $18x+42-8x+28+231=21x-84$;
2d *step*, $18x-8x-21x=-231-42-28-84$;
3d *step*, $-11x=-385$,
$x=35$.

VERIFICATION, $\dfrac{3\times35+7}{14}-\dfrac{2\times35-7}{21}+2\frac{3}{4}=\dfrac{35-4}{4}$,

$$8-3+2\tfrac{3}{4}=7\tfrac{3}{4},$$
$$7\tfrac{3}{4}=7\tfrac{3}{4}.$$

8. $5(x+1)-2=3(x+5)$. . . . . Ans. $x=6$.
9. $3(x-2)+4=4(3-x)$. . . . . Ans. $x=2$.
10. $5-3(4-x)+4(3-2x)=0$. . . . . . Ans. $x=1$.

11. $\frac{x}{2}+\frac{x}{3}=\frac{x}{4}+7$ . . . . Ans. $x=12$.

12. $\frac{x}{2}+\frac{x}{3}-\frac{x}{4}+\frac{x}{5}=7\frac{5}{6}$. . . . . . . Ans. $x=10$.

13. $\frac{1}{x}+\frac{1}{2x}-\frac{1}{3x}=\frac{7}{3}$. . . . . . . Ans. $x=\frac{1}{2}$.

14. $\frac{3x+1}{2}-\frac{2x}{3}=10+\frac{x-1}{6}$. . . . . . Ans. $x=14$.

15. $\frac{x-7\frac{1}{2}}{2}=\frac{3x-9}{4}+\frac{27-5x}{3}$. . . . . Ans. $x=7\frac{7}{17}$.

16. $5x-\frac{2x-1}{3}+1=3x+\frac{x+2}{2}+7$. . Ans. $x=8$.

17. $\frac{7x+9}{8}-\frac{3x+1}{7}=\frac{9x-13}{4}-\frac{249-9x}{14}$. Ans. $x=9$.

18. $\frac{1}{3}(2x-10)-\frac{1}{11}(3x-40)=15-\frac{1}{5}(57-x)$.

Ans. $x=17$.

19. $\frac{1}{4}(4+\frac{3}{2}x)-\frac{1}{7}(2x-\frac{1}{3})=\frac{31}{28}$. . . . . Ans. $x=\frac{2}{3}$.

20. $3\frac{1}{3}\times\left\{28-\left(\frac{x}{8}+24\right)\right\}=3\frac{1}{2}\times\left\{2\frac{1}{3}+\frac{x}{4}\right\}$.

Ans. $x=4$.

21. $\frac{1}{2}(x-\frac{51}{26})-\frac{2}{13}(1-3x)=x-\frac{1}{30}\left(5x-\frac{1-.3x}{.4}\right)$.

Ans. $x=11$.

When one or more of the denominators is a compound quantity, as in the two following examples, it is generally best to multiply all the terms by the L.C.M. of the other denominators, collect the terms, and proceed as before.

22. $\frac{9x+3}{27}+\frac{3x-6}{2x-5}=\frac{2}{3}+\frac{3x+22}{9}$. . . . Ans. $x=3$.

23. $\frac{3x-1}{x+2}+\frac{6+x}{4}-\frac{3x-9}{12}=2\frac{1}{4}+\frac{3x+9}{x+7}$. Ans. $x=5$.

24. $bx+2x-a=3x-2c$. . . . . . . Ans. $x=\frac{a-2c}{b-1}$.

25. $a^2x+b^3=b^2x+a^3$. . . . . . Ans. $x=\frac{a^2+ab+b^2}{a+b}$.

26. $ax+b^2=a^2+bx$. . . . . Ans. $x=a+b$.

27. $\frac{bx}{a}-\frac{d}{c}=\frac{a}{b}-\frac{cx}{d}$. . . . . Ans. $x=\frac{ad}{bc}$.

28 $\frac{a-b}{x-c}=\frac{a+b}{x+2c}$. . . . . . . Ans. $x=\frac{c}{2b}(3a-b)$.

29. $\frac{x}{a}-1-\frac{dx}{c}+3ab=0$. . . . Ans. $x=\frac{ac(1-3ab)}{c-ad}$.

30. $\frac{1}{3}(x-a)-\frac{1}{5}(2x-3b)-\frac{1}{2}(a-x)=10a+11b$.
Ans. $x=25a+24b$.

31. $\frac{1}{ab-ax}+\frac{1}{bc-bx}=\frac{1}{ac-ax}$. Ans. $x=\frac{b(a-b+c)}{a}$.

QUESTIONS PRODUCING SIMPLE EQUATIONS CONTAINING ONLY ONE UNKNOWN QUANTITY.

154. The solution of a problem by algebra consists of two distinct parts:

1st. *Expressing the conditions of the problem in algebraic language; that is, forming the equation.*

2d. *Solving the equation; that is, finding the value of the unknown quantity.*

Sometimes the proposed problem furnishes the equation directly; and sometimes it is necessary, from the conditions given, to deduce others, from which to form it. In the one case, the conditions are said to be *explicit;* in the other, *implied.*

It is impossible to give a precise rule by means of which every question may be readily stated in the form of an equation. The first step is, to understand *fully* the nature of the question. After this, the equation may generally be formed by the following

Rule.—*Denote the required quantity by one of the final letters of the alphabet; then, by means of signs, indicate the same operations that it would be necessary to perform with the answer, to verify it.*

1. Find two numbers such, that their sum shall be 50, and their difference 12.

Let x denote the least of the two required numbers.

Then will . $x+12=$ the greater,

And . . $x+x+12=50$, by the question.

Transposing, . $x+x=50-12$.

Reducing, . $2x=38$.

Dividing, . . $x=19$, the less number;

And $x+12=19+12=31$, the greater number.

VERIFICATION, $31+19=50$, and $31-19=12$.

2. What number is that whose $\frac{1}{3}$ exceeds its $\frac{1}{5}$ by 6?

Let $x=$ the required number.

Then will its $\frac{1}{3}$ part be denoted by $\frac{x}{3}$, and its $\frac{1}{5}$ part, by $\frac{x}{5}$.

Therefore, . . $\frac{x}{3}-\frac{x}{5}=6$.

Clearing, . . . $5x-3x=90$.

Reducing, . $2x=90$.

Dividing, . . . $x=45$, the number required.

VERIFICATION, . $\frac{1}{3}$ of $45=15$, $\frac{1}{5}$ of $45=9$; $15-9=6$.

3. A can perform a piece of work in 6 days, and B in 8 days; in what time will both together finish it?

Let $x=$ time required. Then, since A can perform the work in 6 days, he will perform $\frac{1}{6}$ of it in one day, and in x days $\frac{x}{6}$ of the work. Reasoning in the same way with reference to B,

We have $\frac{x}{6}+\frac{x}{8}=1$, the whole work being expressed by unity.

$8x+6x=48$; or, $14x=48$;

And $x=3\frac{3}{7}$, the number of days.

Or, since $\frac{1}{x}$ will represent the part of the work which both perform in one day, the equation may be more properly stated thus:

$$\frac{1}{6}+\frac{1}{8}=\frac{1}{x}.$$

Then, $8x+6x=48$, as before; and $x=3\frac{3}{7}$.

4. Divide \$500 among A, B, and C, so that B shall have \$20 more than A, and C \$75 more than A.

Let x=A's share; $x+20$=B's; and $x+75$=C's.

Then, . $x+\overline{x+20}+\overline{x+75}=500$, by the question.

Reducing, ... $3x+95=500$.

Subtracting 95 from each side, $3x=405$.

Dividing, $x=135$, A's share; $x+20=155$, B's; $x+75=210$, C's.

VERIFICATION, $135+155+210=500$.

5. A person in play lost a fourth of his money, and then won back \$3; after which he lost a third of what he now had, and then won back \$2; lastly, he lost a seventh of what he then had, and after this found he had but \$12 remaining; what had he at first?

Let . . . $x=$ money he had at first.

Then, . . . $\frac{x}{4}=$ first loss.

Subtracting and adding 3, $\frac{3x}{4}+3=$ had after 1st game.

$\frac{1}{3}$ of the above, or, $\frac{x}{4}+1=$ second loss.

Subtracting and adding 2, $\frac{x}{2}+4=$ had after 2d game.

$\frac{1}{7}$ of the above, or, . $\frac{x}{14}+\frac{4}{7}=$ third loss.

Subtracting, . . . $\frac{3x}{7}+\frac{24}{7}=$ had after 3d game.

Then, ... $\frac{3x}{7}+\frac{24}{7}=12$; from which we find $x=\$20$.

6. Out of a cask of wine which had leaked away $\frac{1}{5}$, 35 gallons were drawn, and then, being gauged, it was $\frac{1}{3}$ full; how much did it hold?

Let $x=$ the number of gallons it held;

Then, $\frac{x}{5}=$ " " " leaked out.

There had been taken away $\frac{x}{5}+35$ gallons.

There remained $x-\left(\frac{x}{5}+35\right)$ gal.; $\therefore x-\left(\frac{x}{5}+35\right)=\frac{x}{3}$.

From which the answer is readily found.

7. A laborer was engaged for 20 days. For each day that he worked, he received 50 cents and his boarding; and for each day that he was idle, he paid 25 cents for his boarding. At the expiration of the time, he received \$4; how many days did he work, and how many days was he idle?

Let $x=$ the number of days he worked;
Then, $20-x=$ " " " " was idle.
Also, $50x=$ wages due for work.
And $25(20-x)=$ the amount to be deducted for boarding.
$\therefore 50x-25(20-x)=400$.

From which the answer is readily found.

8. What two numbers are as 3 to 5, to each of which, if 9 be added, the sums shall be to each other as 6 to 7?

Let $3x=$ the first, and $5x=$ the second number.
Then, $3x+9:5x+9::6:7$.

But in every proportion, the product of the means is equal to the product of the extremes. (RAY'S ARITH., 3d Book, Art. 200.)

Hence, $6(5x+9)=7(3x+9)$.

From which the answer is readily found.

When, as in the above example, two or more unknown quantities have to each other a given ratio,

Assume each of them a multiple of some other unknown quantity, so that they shall have to each other the given ratio.

9. A courier, who traveled at the rate of $31\frac{1}{2}$ miles in 5 hours, was dispatched from a certain city; 8 hours after his departure, another courier was sent to overtake him, who traveled at the rate of $22\frac{1}{2}$ miles in 3 hours. In what time did he overtake the first, and at what distance from the place of departure?

Let $x=$ the number of hours that the second courier travels. Then, since the first courier travels at the rate of $31\frac{1}{2}$ miles in

5 hours; that is, $\frac{63}{10}$ miles in 1 hour, he will travel $\frac{63x}{10}$ miles in x hours; and since he started 8 hours before the second courier, the whole distance traveled by him will be $(8+x)\frac{63}{10}$.

Again, since the second courier travels at the rate of $22\frac{1}{2}$ miles in 3 hours, that is, $\frac{45}{6}$ miles in 1 hour, he will travel $\frac{45}{6}x$ miles in x hours.

But the couriers are together at the end of the time x; therefore, the distance traveled by each must be the same. Hence,

$$\frac{45x}{6}=(8+x)\frac{63}{10};$$ from which the answer is readily found.

10. A smuggler had a quantity of brandy, which he expected would sell for 198 shillings; after he had sold 10 gallons, a revenue officer seized one third of the remainder, in consequence of which, what he sold brought him only 162 shillings. Required the number of gallons he had, and the price per gallon.

Let $x=$ the number of gallons;

Then, $\frac{198}{x}$ is the price per gallon, in shillings; and $\frac{x-10}{3}$ is the quantity seized, the value of which is $198-162=36$ shillings.

$$\therefore \frac{x-10}{3}\times\frac{198}{x}=36;$$ from which the answer is readily found.

11. There are three numbers whose sum is 133; the second is twice the first, and the third twice the second. Required the numbers. Ans. 19, 38, and 76.

12. There are three numbers whose sum is 187; the second is 3 times, and the third $4\frac{1}{2}$ times, the first. Required the numbers. Ans. 22, 66, and 99.

13. There are two numbers, of which the first is $3\frac{1}{2}$ times the second, and their difference is 100. Required the numbers. Ans. 40 and 140.

14. Two numbers are to each other as 3 to 7; if 16 be added to the first and subtracted from the second, the sum will be to the difference as 7 to 3. What are the numbers? Ans. 12 and 28.

15. What two numbers are to each other as 2 to 3, to each of which if 6 be added the sums will be as 4 to 5?

Ans. 6 and 9.

16. A person, at the time of his marriage, was three times as old as his wife, but 15 years after he was only twice as old. What were their ages on their wedding day?

Ans. Man 45, and wife 15.

17. A bill of $34 was paid in half dollars and dimes, and the number of pieces of both sorts was 100; how many were there of each? Ans. 60 half dollars, 40 dimes.

18. There are three numbers whose sum is 156; the second is $3\frac{1}{2}$ times the first, and the third is equal to the remainder left, after subtracting the difference of the first and second from 100. Required the numbers.

Ans. 28, 98, and 30.

19. What number is that, whose half, third, and fourth parts, taken together, are equal to 52? Ans. 48.

20. What number is that, which being increased by its six sevenths, and diminished by 20, shall be equal to 45?

Ans. 35.

21. What number is that, to which if its third and fourth parts be added, the sum will exceed its sixth part by 51?

Ans. 36.

22. Find a number which, being multiplied by 4, becomes as much above 40 as it is now below it. Ans. 16.

23. What number is that, to which if 16 be added, 4 times the sum will be equal to 10 times the number increased by 1? Ans. 9.

24. If a certain number be multiplied by 4, and 20 be added to the product the sum will be 32. What is the number? Ans. 3.

25. If 5 be subtracted from three fourths of a certain number the remainder will be equal to the number divided by 3. Required the number. Ans. 12.

26. The rent of an estate is greater by 8 % than it was last year. The rent this year is \$1890. What was it last year? Ans. \$1750.

Observe that the interest on any sum of money is found by multiplying the principal by the rate per cent., and dividing by 100.

27. An estate is divided as follows: The eldest child receives one fourth, the second 20 %, and the third 15 % of the whole. The remainder, which is \$2168, is given to the widow. Required the value of the estate, and the share of each child.

Ans. Estate \$5420; shares \$1355, \$1084, and \$813.

28. The sum of two numbers is 30; and if the less be subtracted from the greater, one fourth of the remainder will be 3. Required the numbers. Ans. 9 and 21.

29. A laborer was engaged for 28 days, upon the condition that for every day he worked he was to receive 75 cents, and for every day he was absent, he was to forfeit 25 cents. At the end of his time he received \$12. How many days did he work? Ans. 19.

30. At what time between two and three o'clock will the hour and minute hands of a watch be together?

Ans. 2h. 10m. $54\frac{6}{11}$ sec.

The face of a watch is divided into 60 minute spaces, and the minute hand moves twelve times as fast as the hour hand.

Let $x=$ distance from XII to the point of meeting; it will also express the number of min. after 2 when the hands are together.

Let $x=$ No. min. after 2 o'clock, or distance min. hand has gone.

Then, $x-10=$ distance hour hand has gone after 2 o'clock.

$$x=12(x-10)=12x-120;$$

$11x=120$; and $x=10\frac{10}{11}$ min., or the hands are together 10 min. $54\frac{6}{11}$ sec. after 2 o'clock.

31. The hour and minute hand of a clock are together at noon; when are they next together?

Ans. 1h. $5\frac{5}{11}$ min.

32. At what time between 8 and 9 o'clock are the hour and minute hands of a watch opposite to each other?
Ans. 8 h. $10\frac{10}{11}$ min.

33. A has three times as much money as B, but if B give A \$50, then A will have four times as much as B. Find the money of each. Ans. A, \$750; B, \$250.

34. From a bag of money which contained a certain sum, there was taken \$20 more than its half; from the remainder, \$30 more than its third part; and from the remainder, \$40 more than its fourth part, and then there was nothing left. What sum did it contain? Ans. \$290.

35. A merchant gains the first year, 15 % on his capital; the second year, 20 % on the capital at the close of the first; and the third year, 25 % on the capital at the close of the second; when he finds that he has cleared \$1000 50. Required his capital. Ans. \$1380.

36. A is twice as old as B; 22 years ago, he was three times as old. What is A's age? Ans. 88.

37. A person buys 4 houses; for the second, he gives half as much again as for the first; for the third, half as much again as for the second; and for the fourth, as much as for the first and third together: he pays \$8000 for them all. Required the cost of each.
Ans. \$1000, \$1500, \$2250, and \$3250.

38. A cistern is filled in 24 minutes by 3 pipes, the first of which conveys 8 gallons more, and the second 7 gallons less, than the third every 3 minutes. The cistern holds 1050 gallons. How much flows through each pipe in a minute? Ans. $17\frac{5}{36}$, $12\frac{5}{36}$, $14\frac{17}{36}$.

39. A can do a piece of work in 3 days, B in 6 days, and C in 9 days. Find the time in which all together can perform it. Ans. $1\frac{7}{11}$ days.

Let $x=$ the required number of days. Then, in one day, A can do $\frac{1}{3}$, B $\frac{1}{6}$, and C $\frac{1}{9}$, and all three $\frac{1}{x}$ of the whole work.

$$\text{Hence,}\quad \tfrac{1}{3}+\tfrac{1}{6}+\tfrac{1}{9}=\frac{1}{x}.$$

40. If A does a piece of work in 10 days, which A and B can do together in 7 days, how long would it take B to do it alone? Ans. $23\frac{1}{3}$ days.

41. A performs $\frac{2}{7}$ of a piece of work in 4 days; he then receives the assistance of B, and the two together finish it in 6 days. Required the time in which each can do it alone. Ans. A, 14 days; B, 21 days.

42. A person bought an equal number of sheep, cows, and oxen, for \$330; each sheep cost \$3, each cow \$12, and each ox \$18. Required the number of each. Ans. 10.

43. A sum of money is to be divided among five persons—A, B, C, D, and E. B received \$10 less than A; C, \$16 more than B; D, \$5 less than C; E, \$15 more than D; and the shares of the last two are equal to the sum of the shares of the other three. Required the share of each. Ans. A, \$21; B, \$11; C, \$27; D, \$22; E, \$37.

44. A bought eggs at 18 cts. a dozen, but had he bought 5 more for the same money, they would have cost him $2\frac{1}{2}$ cts. a dozen less. How many did he buy? Ans. 31.

45. A person bought a number of sheep for \$94; having lost 7 of them, he sold $\frac{1}{4}$ of the remainder at prime cost, for \$20. How many had he at first? Ans. 47.

46. There are two places, 154 miles distant from each other, from which two persons, A and B, set out at the same instant, to meet on the road. A travels at the rate of 3 mi. in 2 hr., and B at the rate of 5 mi. in 4 hr. How long, and how far, did each travel before they met?

Ans. 56 hr.; A traveled 84, B, 70 mi.

47. A person bought a chaise, horse, and harness, for \$450; the horse came to twice the price of the harness,

and the chaise to twice the price of the horse and harness. What was the cost of each?

Ans. Chaise \$300, horse \$100, harness \$50.

48. There is a fish whose tail weighs 9 lbs.; his head weighs as much as his tail and half his body, and his body weighs as much as his head and his tail. What is his whole weight? Ans. 72 lbs.

49. Find that number, which, multiplied by 5, and 24 taken from the product, the remainder divided by 6, and 13 added to the quotient, will still give the same number.

Ans. 54.

50. In a bag containing eagles and dollars, there are three times as many eagles as dollars; but if 8 eagles and as many dollars be taken away, there will be left five times as many eagles as dollars. How many were there of each?

Ans. 48 eagles, 16 dollars.

51. If 10 apples cost a cent, and 25 pears cost 2 cents, and you buy 100 apples and pears for $9\frac{1}{2}$ cents, how many of each will you have? Ans. 75 apples and 25 pears.

52. Suppose that for every 8 sheep a farmer keeps, he should plow an acre of land, and allow one acre of pasture for every 5 sheep, how many sheep may he keep on 325 acres? Ans. 1000.

53. A person has just 2 hours spare time; how far may he ride in a stage which travels 12 miles an hour, so as to return home in time, walking back at the rate of 4 miles an hour? Ans. 6 miles.

54. If 65 lbs. of sea-water contain 2 lbs. of salt, how much fresh water must be added to these 65 lbs., in order that the quantity of salt contained in 25 lbs. of the new mixture shall be reduced to $\frac{1}{4}$ of a lb.? Ans. 135 lbs.

55. A mass of copper and tin weighs 80 lbs.; and for every 7 lbs. of copper, there are 3 lbs. of tin. How much copper must be added to the mass, that for every 11 lbs. of copper there may be 4 lbs. of tin? Ans. 10 lbs.

56. A merchant maintained himself for three years, at a cost of \$250 a year; and in each of those years augmented that part of his stock which was not so expended, by $\frac{1}{3}$ thereof. At the end of the third year his original stock was doubled. What was that stock? Ans. \$3700.

SIMPLE EQUATIONS CONTAINING TWO UNKNOWN QUANTITIES.

155. When an equation contains two or more unknown quantities, the value of any one of them is entirely dependent on the rest, and can become *known* only when the values of the rest are *given*, or *known*. Thus, in the equation

$$x+y=a,$$

the value of x depends on the values of y and a, and can only become known when they are known; therefore,

To find the value of any unknown quantity, we must obtain a single equation containing it and known quantities.

The method of doing this is termed *elimination*, which may be defined briefly, thus:

Elimination is the process of deducing, from two or more equations containing two or more unknown quantities, a single equation containing only one unknown quantity.

There are three principal methods of elimination:

1st. *Elimination by Substitution.*
2d. *Elimination by Comparison.*
3d. *Elimination by Addition and Subtraction.*

156. Elimination by Substitution consists in finding the value of one of the unknown quantities in one of the equations, and substituting this value in the other equation.

To explain this method, let it be required to find the values of x and y, in the following equations:

$$2x+3y=33, \qquad (1)$$
$$4x+5y=59. \qquad (2)$$

From (1), by transposing $3y$ and dividing by 2, we have

$$x=\frac{33-3y}{2}.$$

Substituting this value of x, instead of x in (2), we have

$$4\left(\frac{33-3y}{2}\right)+5y=59;$$
$$66-6y+5y=59;$$
$$-y=-7;$$
$$y=7;$$
$$x=\frac{33-3\times7}{2}=6.$$

The following is the general form to which two equations of the first degree, containing two unknown quantities, may always be reduced. The signs of the known quantities, a, b, c, etc., may be either plus or minus.

$$ax+by=c, \qquad (1)$$
$$a'x+b'y=c'. \qquad (2)$$

From (1), by transposing by, and dividing by a, we have

$$x=\frac{c-by}{a}.$$

Substituting this value of x in (2), we have

$$a'\left(\frac{c-by}{a}\right)+b'y=c';$$
$$a'c-a'by+ab'y=ac';$$
$$(ab'-a'b)y=ac'-a'c;$$
$$y=\frac{ac'-a'c}{ab'-a'b}.$$

But $x=\dfrac{c-by}{a}=\dfrac{c-b\left(\dfrac{ac'-a'c}{ab'-a'b}\right)}{a}=\dfrac{ab'c-a'bc-abc'+a'bc}{a(ab'-a'b)}$

$$=\frac{b'c-bc'}{ab'-a'b}.$$ Hence,

Rule for Elimination by Substitution.—*Find an expression for the value of one of the unknown quantities in either*

equation, and substitute this value, instead of the same unknown quantity, in the other equation; there will thus be formed a new equation, containing only one unknown quantity.

1. $\left.\begin{array}{l}3x-5y=2,\\2x+7y=22.\end{array}\right\}$ Ans. $x=4$, $y=2$.

2. $\left.\begin{array}{l}5x-3(x-y)=13,\\x-y=4.\end{array}\right\}$ A. $x=5$, $y=1$.

3. $\left.\begin{array}{l}4x=64-3y,\\2x+3y=44.\end{array}\right\}$ Ans. $x=10$, $y=8$.

4. $\left.\begin{array}{l}5x-7y=0,\\4x+3y=8y+3.\end{array}\right\}$ Ans. $x=7$, $y=5$.

5. $\left.\begin{array}{l}ax+by=c-d,\\mx=ny.\end{array}\right\}$ . . . . . Ans. $x=\frac{n(c-d)}{an+bm}$, $y=\frac{m(c-d)}{an+bm}$.

REMARK.—This method is always to be preferred where the value of one of the unknown quantities may be found in terms of the other, as in examples 4 and 5 above.

157. Elimination by Comparison consists in finding the value of the same unknown quantity in two different equations, and then placing these values equal to each other.

To illustrate this method, we will take the same equations as in the preceding article.

$$2x+3y=33, \qquad (1)$$
$$4x+5y=59. \qquad (2)$$

From (1), by transposing and dividing, we have $x=\frac{33-3y}{2}$.

From (2), by transposing and dividing, we have $x=\frac{59-5y}{4}$.

Placing these values of x equal to each other,

$$\frac{59-5y}{4}=\frac{33-3y}{2};$$

$59-5y=66-6y$, by clearing of fractions;

$y=7$, by transposition.

The value of x may be found similarly, by first finding the values of y, and placing them equal to each other. Or, it may generally be found most readily by substitution. Thus,

$$4x+5\times7=59;$$

Whence, $x=\frac{59-35}{4}=6$.

General equations, $ax+by=c$, (1)
$a'x+b'y=c'$. (2)

From (1), by transposing and dividing, $x=\dfrac{c-by}{a}$.

From (2), by transposing and dividing, $x=\dfrac{c'-b'y}{a'}$.

Equating these values of x,

$$\frac{c-by}{a}=\frac{c'-b'y}{a'};$$

$$a'c-a'by=ac'-ab'y, \text{ by clearing of fraction ;}$$
$$(ab'-a'b)y=ac'-a'c, \text{ by transposing;}$$
$$y=\frac{ac'-a'c}{ab'-a'b}.$$

From (1), $y=\dfrac{c-ax}{b}$; from (2), $y=\dfrac{c'-a'x}{b'}$.

Equating these values of y,

$$\frac{c'-a'x}{b'}=\frac{c-ax}{b};$$
$$bc'-a'bx=b'c-ab'x;$$
$$(ab'-a'b)x=b'c-bc';$$
$$x=\frac{b'c-bc'}{ab'-a'b}.\quad \text{Hence,}$$

Rule for Elimination by Comparison.—*Find an expression for the value of the same unknown quantity in each of the given equations, and place these values equal to each other; there will thus be formed a new equation, containing only one unknown quantity.*

1. $\left.\begin{array}{l}3x-2y=9,\\5x+4y=37.\end{array}\right\}$ Ans. $x=5$, $y=3$.

2. $\left.\begin{array}{l}7x+y=10y+7,\\x+y=2y+3.\end{array}\right\}$ Ans. $x=10$, $y=7$.

3. $\left.\begin{array}{l}4x+9y=51,\\8x-13y=9.\end{array}\right\}$ Ans. $x=6$, $y=3$.

4. $\left.\begin{array}{l}mx=ny,\\x+y=a.\end{array}\right\}$ Ans. $x=\dfrac{an}{m+n}$, $y=\dfrac{am}{m+n}$.

5. $\left.\begin{array}{l}ax+by=p,\\cx+dy=q.\end{array}\right\}$ A. $x=\dfrac{bq-dp}{bc-ad}$, $y=\dfrac{aq-cp}{ad-bc}$.

Remark.—This method is generally to be preferred where the equations are *literal*, and sometimes in other cases.

158. Elimination by Addition and Subtraction consists in multiplying or dividing two equations, so as to render the coëfficient of one of the unknown quantities the same in both; and then, by addition or subtraction, causing the terms containing it to disappear.

Taking the same equations as in the preceding articles,

$$2x+3y=33, \quad (1)$$
$$4x+5y=59. \quad (2)$$

It is evident if we multiply (1) by 2, that the coëfficient of x will be the same in the two equations.

$4x+6y=66$ (3), by multiplying (1) by 2.
$4x+5y=59$, (2) brought down.
$y=7$, by subtraction.

If the signs of the coëfficients of x had been *different*, the terms in x would have been canceled by *adding*.

Having obtained the value of y, that of x may be obtained in the same way, or by substitution. Thus,

Multiply (1) by 5, and (2) by 3, and the coëfficients of y will be the same in both.

$10x+15y=165$, (4) by multiplying (1) by 5.
$12x+15y=177$, (5) by multiplying (2) by 3.
$2x=12$, by subtracting (4) from (5).
$x=6$.

Or, by substitution, from either of the original equations. Thus,

From (1) $2x+3\times7=33$;
$2x=33-21=12$;
$x=6$.

General equations, $ax+by=c, \quad (1)$
$a'x+b'y=c'. \quad (2)$

It is evident that we shall render the coëfficients of x the same in both equations, by multiplying (1) by a', and (2) by a.

$aa'x+a'by=a'c$, (3), by multiplying (1) by a';
$aa'x+ab'y=ac'$, (4), by multiplying (2) by a;
$(ab'-a'b)y=ac'-a'c$, by subtracting;

$$y=\frac{ac'-a'c}{ab'-a'b}.$$

The coëfficients of y in the two equations will evidently become equal by multiplying (1) by b', and (2) by b.

$$\begin{array}{rll} ab'x+bb'y=b'c, & (5) & \text{by multiplying (1) by } b'; \\ a'bx+bb'y=bc', & (6) & \text{by multiplying (2) by } b; \\ \hline (ab'-a'b)x=b'c-bc', & & \text{by subtracting;} \\ x=\dfrac{b'c-bc'}{ab'-a'b}. & & \text{Hence,} \end{array}$$

Rule for Elimination by Addition and Subtraction.—**1**. *Multiply or divide the equations, if necessary, so that one of the unknown quantities will have the same coëfficient in both.*

2. *Take the difference, or the sum, of the equations, according as the signs of the equal terms are alike or unlike, and the resulting equation will contain only one unknown quantity.*

Remark.—When the coëfficients of the quantity to be eliminated are prime to each other, multiply each by the other. When the coëfficients are not prime, multiply by such numbers as will produce their L.C.M.

If the equations have fractional coëfficients, they ought to be cleared, before applying the rule.

1. $\left.\begin{array}{l} x+3y=10, \\ 3x+2y=9. \end{array}\right\}$ Ans. $x=1$, $y=3$.

2. $\left.\begin{array}{l} 2x+3y=18, \\ 3x-2y=1. \end{array}\right\}$ Ans. $x=3$, $y=4$.

3. $\left.\begin{array}{l} 2x-9y=11, \\ 3x-12y=15. \end{array}\right\}$ Ans. $x=1$, $y=-1$.

4. $\left.\begin{array}{l} \dfrac{4x+5y}{40}=x-y, \\ \dfrac{2x-y}{3}+2y=\frac{1}{2}. \end{array}\right\}$ A. $x=\frac{1}{4}$, $y=\frac{1}{5}$.

5. $\left.\begin{array}{l} x+ay=b, \\ ax-by=c. \end{array}\right\}$ A. $x=\dfrac{ac+b^2}{a^2+b}$, $y=\dfrac{ab-c}{a^2+b}$.

The following examples may be solved by either of the three methods of elimination:

1. $\left.\begin{array}{l} 9x-4y=8, \\ 13x+7y=101. \end{array}\right\}$ Ans. $x=4$, $y=7$.

2. $\left.\begin{array}{l} x-\frac{1}{7}(y-2)=5, \\ 4y-\frac{1}{3}(x+10)=3. \end{array}\right\}$ A. $x=5$, $y=2$.

3. $\left.\begin{array}{l} \dfrac{x}{3}+\dfrac{y}{5}=8, \\ \dfrac{x}{9}-\dfrac{y}{10}=1. \end{array}\right\}$ Ans. $x=18$, $y=10$.

4. $\left.\begin{array}{l}\dfrac{4}{5+y}=\dfrac{5}{12+x}, \\ 2x+5y=35.\end{array}\right\}$ Ans. $x=2$. $y=\frac{31}{5}$.

5. $\left.\begin{array}{l}\dfrac{2x+6}{3y+2}=\frac{8}{7}, \\ 8x-4=9y.\end{array}\right\}$ Ans. $x=5$, $y=4$.

6. $\left.\begin{array}{l}\frac{1}{3}(x+y)+\frac{1}{4}(x-y)=59, \\ 5x-33y=0.\end{array}\right\}$ . . . . Ans. $x=99$, $y=15$.

7. $\left.\begin{array}{l}\dfrac{3x+4y+3}{10}-\dfrac{2x+7-y}{15}=5+\dfrac{y-8}{5}, \\ \dfrac{9y+5x-8}{12}-\dfrac{x+y}{4}=\dfrac{7x+6}{11}.\end{array}\right\}$ . . . . . . Ans. $x=7$, $y=9$.

8. $\left.\begin{array}{l}ax=by, \\ x+y=c.\end{array}\right\}$ . . . . Ans. $x=\dfrac{bc}{a+b}$, and $y=\dfrac{ac}{a+b}$.

9. $\left.\begin{array}{l}3ax-2by=c, \\ a^2x+b^2y=5bc.\end{array}\right\}$ Ans. $x=\dfrac{11bc}{2a^2+3ab}$, and $y=\dfrac{c}{b}\left(\dfrac{15b-a}{2a+3b}\right)$.

10. $\left.\begin{array}{l}\dfrac{m}{x}+\dfrac{n}{y}=a, \\ \dfrac{n}{x}+\dfrac{m}{y}=b.\end{array}\right\}$ . . . . Ans. $x=\dfrac{m^2-n^2}{ma-nb}$, $y=\dfrac{m^2-n^2}{mb-na}$.

11. $\left.\begin{array}{l}(a^2-b^2)(5x+3y)=(4a-b)2ab, \\ a^2y-\dfrac{ab^2c}{a+b}+(a+b+c)bx=b^2y+(a+2b)ab.\end{array}\right\}$ Ans. $x=\dfrac{ab}{a+b}$, $y=\dfrac{ab}{a-b}$.

REMARK.—Transpose b^2y in (2), multiply by 3, and subtract; there will then result an equation involving x.

PROBLEMS PRODUCING SIMPLE EQUATIONS CONTAINING TWO UNKNOWN QUANTITIES.

159. The questions contained in Art. 154, may all be solved by using one unknown symbol, although, in some cases, there were two or more unknown quantities.

It frequently happens, however, that the conditions of a problem are such as to require the use of two or more symbols for the unknown quantities. In this case, the number of equations must be equal to the number of symbols, and

the value of the unknown quantities may be found by either of the three methods of elimination.

A problem may often be solved by using either one or more unknown quantities. In illustration, take the following:

1. The difference of two numbers is a, and the less is to the greater as m to n; required the numbers.

Solution by using one unknown quantity.

Let $mx=$ the less number, and $nx=$ the greater.

Then, $nx-mx=a$.

$$x=\frac{a}{n-m};\quad \therefore mx=\frac{ma}{n-m},\ \text{and } nx=\frac{na}{n-m}.$$

Solution by using two unknown quantities.

Let $x=$ the less number, and $y=$ the greater.

Then, $y-x=a$, (1)

And $x:y::m:n$; or, $my=nx$. (2)

Since $my=nx$, we have $y=\frac{nx}{m}$;

Substituting this value of y in (1), we find as before,

$$x=\frac{ma}{n-m},\ \text{and } y=\frac{na}{n-m}.$$

2. The hour and minute hands of a watch are opposite at 6 o'clock; when are they next opposite?

Let $x=$ minute spaces moved over by the hour hand, and $y=$ minute spaces moved over by the minute hand. Then, since the minute hand moves 12 times as fast as the hour hand,

$$x:y::1:12,\ \text{or } y=12x. \quad (1)$$

But the minute hand must evidently pass over 60 minutes more than the hour hand; hence,

$$y=x+60. \quad (2)$$

Substituting, $12x=x+60$,

$11x=60$,

$x=5\frac{5}{11}$ min.

$y=65\frac{5}{11}$ min. $=1$ h., $5\frac{5}{11}$ m.

Hence, the hands are next opposite at $5\frac{5}{11}$ m. past 7.

In a similar manner the period of coincidence of the hands may be found.

3. There is a number consisting of two digits, which divided by the sum of its digits, gives a quotient 7; but if the digits be written in an inverse order, and the number thence arising be divided by the sum of the digits $+4$, the quotient $=3$. Required the number. Ans. 84.

In solving questions of this kind, observe that any number consisting of two places of figures, is equal to 10 times the figure in the ten's place plus the figure in the unit's place. Thus, 35 is equal to $10\times3+5$; $456=100\times4+10\times5+6$.

Let $x=$ the tens' digit, and $y=$ the units' digit.

Then, $10x+y=$ the number.

And $10y+x=$ the number when the digits are reversed.

$$\text{Also, } \frac{10x+y}{x+y}=7. \qquad \frac{10y+x}{x+y+4}=3.$$

From these equations we readily find $x=8$, and $y=4$.

4. A farmer sells to one man 5 sheep and 7 cows for \$111, and to another, at the same rate, 7 sheep and 5 cows for \$93. Required the price of a sheep and of a cow.

Ans. Sheep, \$4; cow, \$13.

5. If 7 lbs. of tea and 9 lbs. of coffee cost \$5.20, and, at the same rate, 4 lbs. of tea and 11 lbs. of coffee cost \$3.85, what is the price of a lb. of each?

Ans. Tea, 55c.; coffee, 15c.

6. A and B are in trade together with different sums; if \$50 be added to A's money, and \$20 be taken from B's, they will have the same sum; but if A's money were 3 times, and B's 5 times as great as each really is, they would together have \$2350. How much has each?

Ans. A, \$250; B, \$320.

7. A and B together have \$9800; A invests the sixth part of his money in business, and B the fifth part, and then each has the same sum remaining. How much has each? Ans. A, \$4800; B, \$5000.

SUGGESTION.—Let $6x=$ A's money, and $5y=$ B's.

8. Find a fraction, such that if the numerator and denominator be each increased by 1, the value is $\frac{1}{2}$; but if each be diminished by 1, the value is $\frac{1}{3}$. Ans. $\frac{3}{7}$.

9. Find two numbers, such that $\frac{1}{3}$ of the first exceeds $\frac{1}{4}$ of the second by 3, and $\frac{1}{4}$ of the first and $\frac{1}{5}$ of the second are together equal to 10. Ans. 24 and 20.

10. A grocer knows neither the weight nor the first cost of a box of tea he had purchased. He only recollects that if he had sold the whole at 30 cts. per lb , he would have gained $1, but if he had sold it at 22 cts. per lb., he would have lost $3. Required the number of lbs. in the box, and the first cost per lb. Ans. 50 lbs. at 28 cts.

11. The rent of a field is a certain fixed number of bu. of wheat, and a fixed number of bu. of corn. When wheat is 55 cts., and corn 33 cts. per bu., the portions of rent by wheat and corn are equal; but when wheat is 65 cts. and corn 41 cts., the rent is increased by $1,40. What is the grain rent? Ans. 6 bu. of wheat, 10 of corn.

12. The quantity of water which flows from an orifice is proportional to the area of the orifice, and the velocity of the water. Now, there are two orifices, the areas being as 5 to 13, and the velocities as 8 to 7; and from one there issued in a certain time 561 cubic feet more than from the other. How much water did each discharge?

Ans. 440 and 1001 cubic feet.

13. Find two numbers in the ratio of 5 to 7, to which two other required numbers, in the ratio of 3 to 5, being respectively added, the sums shall be in the ratio of 9 to 13; and the difference of those sums $=16$.

Ans. 30 and 42, 6 and 10.

14. A boy spends 30 cts. in apples and pears, buying his apples at 4 and his pears at 5 for a ct.; he then finds that half his apples and $\frac{1}{3}$ of his pears cost 13 cts. How many of each did he buy? Ans. 72 apples, 60 pears.

15. A farmer rents a farm for \$245 per year; the tillable land being valued at \$2 per acre, and the pasture at \$1.40; now the number of acres of tillable, is to half the excess of the tillable above the pasture, as 28 to 9. How many acres are there of each?

Ans. 98 acres tillable, 35 of pasture.

16. Find that number of 2 figures to which, if the number formed by changing the places of the digits be added, the sum is 121; and if the less of the same two numbers be taken from the greater, the remainder is 9. Ans. 65 or 56.

17. To determine three numbers such that if 6 be added to the first and second, the sums will be in the ratio of 2 to 3; if 5 be added to the first and third, the sums will be in the ratio of 7 to 11; but if 36 be subtracted from the second and third, the remainders will be as 6 to 7.

Ans. 30, 48, 50.

SUGGESTION.—Let $2x-6$, $3x-6$, and y be the numbers.

18. Two persons, A and B, can perform a piece of work in 16 days. They work together for 4 days, when A, being called off, B is left to finish it, which he does in 36 days more. In what time could each do it separately?

Ans. A in 24, B in 48 days.

19. A and B drink from a cask of beer for 2 hr., after which A falls asleep, and B drinks the remainder in 2 hr. and 48 min.; but if B had fallen asleep and A had continued to drink, it would have taken him 4 hr. and 40 min. to finish the cask. In what time could each singly drink the whole? Ans. A in 10, B in 6 hrs.

20. Divide the fraction $\frac{8}{5}$ into two parts, so that the numerators of the two parts taken together shall be equal to their denominators taken together. Ans. $\frac{1}{2}$ and $\frac{11}{10}$.

21. A purse holds 19 crowns and 6 guineas. Now 4 crowns and 5 guineas fill $\frac{17}{63}$ of it. How many of each will it hold? Ans. 21 crowns or 63 guineas.

22. When wheat was 5 shillings a bu. and rye 3 shillings, a man wanted to fill his sack with a mixture of rye and wheat for the money he had in his purse. If he bought 7 bu. of rye and laid out the rest of his money in wheat, he would want 2 bu. to fill his sack; but if he bought 6 bu. of wheat, and filled his sack with rye, he would have 6 shillings left. How must he lay out his money, and fill his sack?

Ans. Buy 9 bu. of wheat, and 12 rye.

SIMPLE EQUATIONS, INVOLVING THREE OR MORE UNKNOWN QUANTITIES.

160. Simple equations, involving three or more unknown quantities, may be solved by either of the three methods of elimination, explained in Arts. 155 to 159; but the third method is generally to be preferred.

1. Given $5x-4y+2z=48$, (1)
$3x+3y-4z=24$, (2)
$2x-5y+3z=19$, (3) to find x, y, and z.

To eliminate z from the first two equations, multiply (1) by 2, and then add this to (2), thus,

$$\begin{array}{rl} 10x-8y+4z=\ 96, & \text{by multiplying (1) by 2,} \\ 3x+3y-4z=\ 24, & (2) \\ \hline 13x-5y\qquad =120, & \text{(5) by adding.} \end{array}$$

To eliminate z from equations (1) and (3), multiply (1) by 3, and (3) by 2, and then subtract; thus,

$$\begin{array}{rl} 15x-12y+6z=144, & \text{by multiplying (1) by 3,} \\ 4x-10y+6z=\ 38, & \text{by multiplying (3) by 2,} \\ \hline 11x-\ 2y\qquad =106, & \text{(6) by subtracting.} \end{array}$$

To eliminate y from equations (5) and (6), multiply (5) by 2, and (6) by 5, and then subtract; thus,

$$\begin{array}{rl} 55x-10y=530, & \text{by multiplying (6) by 5,} \\ 26x-10y=240, & \text{by multiplying (5) by 2,} \\ \hline 29x\qquad =290; & \text{by subtracting.} \\ x=10. & \end{array}$$

$110-2y=106$, by substituting 10 for x in (6); whence, $y=2$.
$50-8+2z=48$, by substituting for x and y in (1); whence, $z=3$.

It is evident that the same method may be applied when the number of equations is four or more. Hence,

General Rule for Elimination by Addition and Subtraction.—1st. *Combine any one of the equations with each of the others, so as to eliminate the same unknown quantity; the number of equations and of unknown quantities will be one less.*

2d. *Combine any one of these new equations with each of the others, as before; the number of equations and of unknown quantities will be two less.*

3d. *Continue this series of operations until a single equation is obtained, with one unknown quantity, and find its value.*

4th. *Substitute this value in the derived equations, for the values of the other unknown quantities.*

REMARK.—In some particular instances, solutions may be obtained more easily and elegantly by other means. As specimens, we present the following:

2. Given
$$\begin{aligned} -x+y+z&=a, \quad (1)\\ x-y+z&=b, \quad (2)\\ x+y-z&=c, \quad (3) \end{aligned}$$
to find x, y, and z.

By adding the three equations together, and calling $a+b+c=s$,

We find $\qquad x+y+z=s. \quad (4)$

Then, by subtracting (1), (2), and (3), respectively from (4), and dividing by 2,

We find . . .
$$\begin{aligned} x&=\tfrac{1}{2}(s-a), \quad =\tfrac{1}{2}(b+c).\\ y&=\tfrac{1}{2}(s-b), \quad =\tfrac{1}{2}(a+c).\\ z&=\tfrac{1}{2}(s-c), \quad =\tfrac{1}{2}(a+b). \end{aligned}$$

In a similar manner, solve the following examples:

3. $\left.\begin{aligned} x+y+z&=22,\\ y+z+u&=21,\\ x+z+u&=19,\\ x+y+u&=16, \end{aligned}\right\}$ . . . . Ans. $x=5$, $y=7$, $z=10$, $u=4$.

4. $\left.\begin{aligned} 2x-y-z&=12, \quad (1)\\ 3y-x-z&=16, \quad (2)\\ 5z-x-y&=24. \quad (3)\end{aligned}\right\}$

Put $x+y+z=s$, and add (1), (2), and (3) successively to the last equation.

This gives . . . . . $3x=s+12$ (a)
$4y=s+16$ (b)
$6z=s+24$ (c)

Multiplying these by 4, 3, and 2, we have

$$\begin{aligned} 12x&=4s+\ 48\\ 12y&=3s+\ 48\\ 12z&=2s+\ 48\end{aligned}$$

$$12(x+y+z)=9s+144, \text{ by addition.}$$

Or, . . $12s=9s+144;$

$$3s=144;$$
$$s=\ 48.$$

Substituting this value of s in (a), (b), and (c), we find $x=20$, $y=16$, and $z=12$.

Solve the following by either method of elimination:

5. $\left.\begin{aligned} x+y+z&=6,\\ 3x-y+2z&=7,\\ 4x+3y-z&=7,\end{aligned}\right\}$ . . . . Ans. $x=1$, ... $y=2$, ... $z=3$.

6. $\left.\begin{aligned} 3x+4y-5z&=32,\\ 4x-5y+3z&=18,\\ 5x-3y-4z&=\ 2.\end{aligned}\right\}$ . . . . . Ans. $x=10$, . . . $y=\ 8$, ... $z=\ 6$.

7. $\left.\begin{aligned} x-9y+3z-10u&=21,\\ 2x+7y-z-u&=683,\\ 3x+y+5z+2u&=195,\\ 4x-6y-2z-9u&=516.\end{aligned}\right\}$ . . . . Ans. $x=100$, ... $y=60$, ... $z=-13$, ... $u=-50$.

8. $\left.\begin{aligned} x+\tfrac{1}{2}y&=10-\tfrac{1}{3}z,\\ \tfrac{1}{2}(x+z)&=9-y,\\ \tfrac{1}{4}(x-z)&=2y-7.\end{aligned}\right\}$ . . . . Ans. $x=7$, ... $y=4$, ... $z=3$.

9. $\left.\begin{array}{l} 9x-2z+u=41, \\ 7y-5z-t=12, \\ 4y-3x+2u=5, \\ 3y-4u+3t=7, \\ \quad 7z-5u=11. \end{array}\right\}$ . . . . Ans. $x=5$, $y=4$, $z=3$, $u=2$, $t=1$.

Examples to be solved by special methods:

10. $\left.\begin{array}{l} \frac{1}{x}+\frac{1}{y}=a, \\ \frac{1}{x}+\frac{1}{z}=b, \\ \frac{1}{y}+\frac{1}{z}=c, \end{array}\right\}$ . . . . Ans. $x=\frac{2}{a+b-c}$, $y=\frac{2}{a-b+c}$, $z=\frac{2}{b+c-a}$.

SUGGESTION.—Subtract (3) from (2), then combine the resulting equation with (1), to find x and y; z may be found similarly.

11. $\left.\begin{array}{l} -x+y+z+v=a, \\ x-y+z+v=b, \\ x+y-z+v=c, \\ x+y+z-v=d, \end{array}\right\}$ . . . . . Ans. $x=\frac{1}{2}(s-a)$, $y=\frac{1}{2}(s-b)$, $z=\frac{1}{2}(s-c)$, $v=\frac{1}{2}(s-d)$,

where $s=\frac{1}{2}(a+b+c+d)$.

PROBLEMS PRODUCING SIMPLE EQUATIONS CONTAINING THREE OR MORE UNKNOWN QUANTITIES.

161. For the method of forming the equations, see Arts. 154 and 159.

1. The stock of three traders amounts to \$760; the shares of the first and second exceed that of the third by \$240; and the sum of the second and third exceeds the first by \$360; what is the share of each?

Ans. \$200, \$300, and \$260.

2. What three numbers are there, each greater than the preceding, whose sum is 20, and such that the sum of the first and second is to the sum of the second and third,

as 4 is to 5; and the difference of the first and second, is to the difference of the first and third, as 2 to 3?

Ans. 5, 7, and 8.

3. Find four numbers, such that the sum of the first, second, and third shall be 13; the sum of the first, second, and fourth, 15; the sum of the first, third, and fourth, 18; and lastly, the sum of the second, third, and fourth, 20.

Ans. 2, 4, 7, 9.

4. The sum of three digits composing a certain number is 16; the sum of the left and middle digits, is to the sum of the middle and right ones as 3 to $3\frac{2}{3}$; and if 198 be added to the number, the order of the digits will be inverted. Required the number. Ans. 547.

5. It is required to find three numbers such that $\frac{1}{2}$ the first, $\frac{1}{3}$ the second, and $\frac{1}{4}$ the third, shall together make 46; $\frac{1}{3}$ the first, $\frac{1}{4}$ the second, and $\frac{1}{5}$ the third, shall together make 35; and $\frac{1}{4}$ the first, $\frac{1}{5}$ the second, and $\frac{1}{6}$ the third, shall together make $28\frac{1}{3}$. Ans. 12, 60, and 80.

6. The sum of three numbers, taken two and two, are a, b, and c. What are the numbers?

Ans. $\frac{1}{2}(a+b-c)$, $\frac{1}{2}(a+c-b)$, and $\frac{1}{2}(b+c-a)$.

7. A person has four casks, the second of which being filled from the first, leaves the first $\frac{4}{7}$ full. The third being filled from the second, leaves it $\frac{1}{4}$ full; and when the third is emptied into the fourth, it is found to fill only $\frac{9}{16}$ of it. But the first will fill the third and fourth and have fifteen quarts remaining. How many quarts does each hold?

Ans. 140, 60, 45, and 80, respectively.

8. In the crew of a ship consisting of sailors and soldiers, there were 22 sailors to every 3 guns, and 10 sailors over; also the whole number of hands was 5 times the number of soldiers and guns together; but after an engagement, in which the slain were one fourth of the survivors, there wanted 5 men to be 13 men to every 2 guns. Required the number of guns, soldiers, and sailors.

Ans. 90 guns, 55 soldiers, 670 sailors.

V. SUPPLEMENT TO SIMPLE EQUATIONS.

REMARK.—The principles employed in algebraic equations may be variously applied. We may, for example, by their aid demonstrate several of the theorems in fractions. Thus, to prove that $\frac{ma}{mb}=\frac{a}{b}$, (Art. 118); put $q=\frac{a}{b}$.

Then, $bq=a$, and $mbq=ma$; $\therefore q=\frac{ma}{mb}$.

Hence, since $q=\frac{a}{b}$, and $q=\frac{ma}{mb}$, $\therefore \frac{ma}{mb}=\frac{a}{b}$.

To prove that $\frac{a}{b}\times\frac{c}{d}=\frac{ac}{bd}$, (Art. 131); put $p=\frac{a}{b}$ and $q=\frac{c}{d}$.

Then, $bp=a$, and $dq=c$. Multiplying the last two equations, member by member,

We have $bdpq=ac$; $\therefore pq=\frac{ac}{bd}$, which proves the rule.

In a similar manner, the rules for Addition, Subtraction, and Division of fractions may be demonstrated.

Other methods of application are given in Arts. following.

I. GENERALIZATION.

162. Literal Equations are those in which the known quantities are represented, either entirely or partly, by letters.

Quantities represented by letters are termed *general* values, because the solution of one problem furnishes a *general* solution.

A **Formula** is the answer to a problem, when the known quantities are represented by letters.

A **Rule** is a formula expressed in ordinary language.

By the application of Algebra to the solution of *general* questions, many useful and interesting truths and rules may be established. Take the following as an example:

163. Divide a given number a into three parts, having to each other the same ratio as the numbers m, n, and p.

Let mx, nx, and px, represent the required parts.

Then, $$mx+nx+px=a,$$

And . . . . $x=\frac{a}{m+n+p}$; from which we obtain,

$$mx=\frac{ma}{m+n+p},\quad nx=\frac{na}{m+n+p},\ \text{and}\ px=\frac{pa}{m+n+p}$$

This formula, expressed in words, gives the following

Rule for Dividing a Given Number into Parts having to each other a Given Ratio.—*Multiply the given number by each term of the ratios respectively, and divide the products by the sum of the numbers expressing the ratios.*

Solve the following examples by this rule, and test its accuracy by verifying the results:

2. Divide 69 into three parts, having to each other the same ratio as the numbers 5, 7, and 11.

Ans. 15, 21, and 33.

3. Divide $38\frac{1}{2}$ into four parts, having to each other the same ratio as the fractional numbers $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, and $\frac{1}{5}$.

Ans. 15, 10, $7\frac{1}{2}$, and 6.

Solve the following general examples, express the formula in ordinary language, so as to form a general rule, and apply the rule or the formula, to the solution of the numerical problems.

4. The sum of two numbers is a, and their difference b. Required the numbers.

A. Greater, $\frac{a}{2}+\frac{b}{2}$; less, $\frac{a}{2}-\frac{b}{2}$.

5. The joint capital of A and B in a firm, is \$16000; but A's investment is \$2000 more than B's. Required the capital of each. Ans. A's, \$9000; B's, \$7000.

6. The sum of two angles is 120° 44′ 52″, and their difference is 26° 32′ 18″ Required the angles.

Ans. Greater, 73° 38′ 35″; less, 47° 6′ 17″.

7. The difference of two numbers is a, and the greater is to the less as m to n. Find the numbers.

Ans. $\frac{ma}{m-n}$, $\frac{na}{m-n}$.

8. The difference in capacity of two cisterns is 678 gal., and the greater is to the less as 7 to 5. How much does each hold? Ans. Greater, 2373 gal.; less, 1695.

9. The sum of two numbers is a, and their sum is to their difference as m to n. Required the numbers.

Ans. Greater, $=\frac{(m+n)a}{2m}$; less, $\frac{(m-n)a}{2m}$.

10. An estate, valued at \$8745, was divided between a son and daughter in such a manner that the sum of their shares was to the difference as 5 to 3. What was the share of each? Ans. Son's, \$6996; daughter's, \$1749.

11. Divide the number a into three such parts, that the second shall exceed the first by b, and the third exceed the second by c. Ans. $\frac{a-2b-c}{3}$, $\frac{a+b-c}{3}$, $\frac{a+b+2c}{3}$.

12. At a certain election the whole number of votes cast was 602. B received 84 more votes than A, and C 56 more than B. How many did each receive?

Ans. A 126, B 210, C 266.

13. Divide the number a into four such parts, that the first increased by m, the second diminished by m, the third multiplied by m, and the fourth divided by m, shall be all equal to each other.

Ans. $\frac{ma}{(m+1)^2}-m$, $\frac{ma}{(m+1)^2}+m$, $\frac{a}{(m+1)^2}$, $\frac{m^2a}{(m+1)^2}$.

Let the four parts be represented by $x-m$, $x+m$, $\frac{x}{m}$, and mx.

14. Divide the number 245 into four parts, such that the first increased by 6, the second diminished by 6, the third multiplied by 6, and the fourth divided by 6, shall be all equal to each other. Ans. 24, 36, 5, and 180.

15. A person has just a hours at his disposal; how far may he ride in a coach which travels b miles an hour, so as to return home in time, walking back at the rate of c miles an hour? Ans. $\frac{abc}{b+c}$ miles.

16. A person finds that he can row a skiff 6 miles an hour with the current, and 3 miles an hour against it; how far can he pass down the stream, and yet return to the point from which he set out, in 8 hours? Ans. 16 miles.

17. Given the sum of two numbers $=a$, and the quotient of the greater divided by the less $=b$. Required the numbers.

Ans. Less $=\frac{a}{b+1}$, greater $=\frac{ab}{b+1}$.

This gives the following simple rule: *To find the less number, divide the sum of the numbers by their quotient increased by unity.*

18. The sum of two numbers is 256, and the quotient of the greater by the less is 15. Required the numbers.

Ans. 240 and 16.

19. A person distributed a cents among n beggars, giving b cents to some, and c to the rest. How many were there of each? Ans. $\frac{a-nc}{b-c}$ at b cts., and $\frac{nb-a}{b-c}$ at c cts.

20. A father divided \$8500 among 7 children, giving to each son \$1750, and to each daughter \$500. How many of his children were sons and how many daughters?

Ans. 4 sons, 3 daughters.

21. Divide the number n into two such parts, that the quotient of the greater divided by the less shall be q, with a remainder r.

Ans. $\dfrac{nq+r}{1+q}$, $\dfrac{n-r}{1+q}$.

22. Divide 1903 into two such parts that the quotient of the greater divided by the less shall be 12, with a remainder 5. Ans. 1757 and 146.

23. If A and B together can perform a piece of work in a days, A and C in b days, and B and C in c days: find the time in which each can perform it separately.

Ans. A in $\dfrac{2abc}{ac+bc-ab}$, B in $\dfrac{2abc}{ab+bc-ac}$, C in $\dfrac{2abc}{ab+ac-bc}$ da.

24. A tank is supplied with water from three pumps. The first and second will fill it in 30 hours, the first and third in 40 hours, and the second and third in 50 hours. In what time can each fill it separately?

Ans. First in $52\frac{4}{23}$, second in $70\frac{10}{17}$, third in $171\frac{3}{7}$ hrs.

25. A, B, and C hold a pasture in common, for which they pay P \$ a year. A puts in a oxen for m months; B, b oxen for n months; and C, c oxen for p months. Required each one's share of the rent.

Ans. A's, $\dfrac{ma}{ma+nb+pc} P$ \$; B's, $\dfrac{nb}{ma+nb+pc} P$ \$; and

C's, $\dfrac{pc}{ma+nb+pc} P$ \$.

From these formulas is derived the rule of *Compound Fellowship.*

26. A, B, and C engage in business together. A put into the firm \$600 for 30 weeks, B \$500 for 40 weeks, and C \$800 for 28 weeks. They then divided a profit of \$1812 between them. What was each man's share?

Ans. A's, \$540; B's, \$600; C's, \$672.

27. A mixture is made of a lb. of tea at m shillings per

lb., b lb. at n shillings, and c lb. at p shillings: what will be its cost per lb.? Ans. $\frac{ma+nb+pc}{a+b+c}$.

From this formula is derived the rule termed *Alligation Medial.*

28. A drover bought 10 cattle at \$30 apiece, 12 at \$40, and 8 at \$90. What was the average price per head? Ans. \$50.

29. A waterman rows a given distance a and back again in b hours, and finds that he can row c miles with the current for d miles against it: required the times of rowing down and up the stream, also the rate of the current and the rate of rowing.

Ans. Time down, $\frac{bd}{c+d}$; time up, $\frac{bc}{c+d}$; rate of current, $\frac{a(c^2-d^2)}{2bcd}$; rate of rowing, $\frac{a(c+d)^2}{2bcd}$.

30. A vessel sailed *with* the wind and tide 60 miles, and returned *with* the wind and *against* the tide. She reached the same point in 12 hours, and the rate of sailing out and in was as 5 to 3. Required the time each way, and the strength of the wind and tide.

Ans. Time out, $4\frac{1}{2}$ hours; time in, $7\frac{1}{2}$ hours; wind, $10\frac{2}{3}$ miles per hour; tide, $2\frac{2}{3}$ miles per hour.

II. NEGATIVE SOLUTIONS.

164. It sometimes happens, in the solution of a problem, that the result has the *minus* sign. This is termed a negative solution. We shall now examine a question of this kind.

1. What number must be subtracted from 3 that the remainder may be 7?

Let $x=$ the number

Then, $3-x=7$; whence, $-x=7-3$; or, $x=-4$.

Now, -4 subtracted from 3, gives a remainder 7; and the answer, -4, is said to satisfy the question in an *algebraic sense.*

The problem is evidently impossible in an *arithmetical sense,* and this *impossibility* is shown by the *negative* answer. But, since subtracting -4 is the same as adding $+4$ (Art. 48), the result is the answer to the following:

What number must be *added* to 3, that the *sum* may be equal to 7?

Let the question now be generalized, thus:

What number must be subtracted from a, that the remainder may be b?

Let . . . . $x=$ the number.

Then, $a-x=b$; whence, $x=a-b$.

While b is *less* than a, the value of x will be *positive;* and the question will be consistent in an *arithmetical sense.*

But if b becomes greater than a, the value of x will be *negative;* and the question will be consistent in its *algebraic,* but not in its *arithmetical sense.*

When b becomes greater than a, the question, to be consistent in an *arithmetical sense,* should read:

What number must be *added* to a that the *sum* may be equal to b?

From this we derive the following important general principles:

1st. *A negative solution indicates some arithmetical inconsistency or absurdity in the question from which the equation was derived.*

2d. *When a negative solution is obtained, the question, to which it is the answer, may be so modified as to be consistent with arithmetical notions.*

After solving the following questions, let them be so modified that the results may be true in an arithmetical sense.

2. What number must be *added* to the number 30, that the *sum* may be 19? ($x=-11$).

3. The *sum* of two numbers is 9, and their *difference* 25; required the numbers. Ans. 17 and -8.

4. What number is that whose half subtracted from its third leaves a remainder 15? ($x=-90$).

5. A father's age is 40 years; his son's age is 13 years; *in how many years* will the age of the father be 4 times that of the son? ($x=-4$).

III. DISCUSSION OF PROBLEMS.

165. After a question has been *generalized* and solved, we may inquire what values the results will have, when particular suppositions are made with regard to the known quantities.

The determination of these values, and the examination of the various results, to which different suppositions give rise, constitute the *discussion of the problem.*

The various forms which the value of the unknown quantity may assume, are shown in the discussion of the following:

1. After subtracting b from a, what number, multiplied by the remainder, will give a product equal to c?

Let $x=$ the number.

Then, $(a-b)x=c$, and $x=\frac{c}{a-b}$.

This result may have five different forms, depending on the different values that may be given to a, b, and c.

To express these forms, let A denote, indefinitely, *some quantity.*

I. When b is less than a. In this case, since $a-b$ will be positive, the value of x will be of the form $+A$.

II. When b is greater than a. In this case, $a-b$ will be negative, and the value of x, of the form $-A$.

III. When b is equal to a. In this case, the value of x is of the form $\frac{A}{0}$, or, (Art. 136), $x=\infty$.

IV. When c is 0, and b either greater or less than a. In this case, the value of x is of the form $\frac{0}{A}$, or, (Art. 136), $x=0$.

V. When b is equal to a, and c is equal to 0. In this case, the value of x is of the form $\frac{0}{0}$, which (Art. 137) is the symbol of indetermination.

The discussion of the following problem, originally proposed by Clairaut, will serve to illustrate further the preceding principles, and show that the results of every correct solution correspond to the circumstances of the problem.

PROBLEM OF THE COURIERS.

166. Two couriers depart at the same time, from two places, A and B, distant a miles from each other; the former travels m miles an hour, and the latter n miles: where will they meet?

There are two cases of this problem, according as the couriers travel *toward* each other, or in the *same* direction.

I. When the couriers travel toward each other.

Let P be the point where they meet, and a=AB, the distance between the two places.

Let x=AP, the distance which the first travels.

Then, $a-x$=BP, the distance which the second travels.

But the distance each travels, divided by the number of miles traveled per hour, will give the number of hours he was traveling.

Therefore, $\frac{x}{m}=$ the number of hours the first travels.

And $\frac{a-x}{n}=$ " " " " second travels.

But they both travel the same number of hours; therefore,

$$\frac{x}{m}=\frac{a-x}{n};$$

$$nx=ma-mx;$$

$$x=\frac{ma}{m+n};\text{ and } a-x=\frac{na}{m+n}.$$

1st. Suppose $m=n$; then, $x=\frac{ma}{2m}=\frac{a}{2}$; and $a-x=\frac{a}{2}$; that is, if they travel at the same rate, each travels half the distance.

2d. Suppose $n=0$; then, $x=\frac{ma}{m}=a$; that is, if the second courier remains at rest, the first travels the whole distance from A to B. In like manner, if $m=0$, $a-x=a$.

3d. Suppose $m>n$, then the value of x will be greater than that of $a-x$, since ma is greater than na; that is, the point P will be farther from A than B If $m<n$, then the value of x will be less than that of $a-x$, or P will be nearer A.

All of these results are evidently true, and correspond to the circumstances of the problem.

II. When the couriers travel in the same direction.

As in the first case, let P be the point of meeting, each traveling from A toward P, and let $a=$AB, the distance between the places;

$x=$AP, the distance the first travels;

$x-a=$BP, the distance the second travels.

Then, reasoning as in the first case, we have

$$\frac{x}{m}=\frac{x-a}{n};$$

$$nx=mx-ma;$$

$$x=\frac{ma}{m-n};\text{ and } x-a=\frac{na}{m-n}.$$

1st. If we suppose m greater than n, the values of x and of $x-a$ will both be positive; that is, the couriers will meet on the right of both A and B. This evidently corresponds to the circumstances of the problem.

2d. If we suppose n greater than m, the value of x, and also that of $x-a$, will be negative.

Now, since the positive values of x and $x-a$ implied that the couriers met at a point P, *on the right* of A and B, the negative values must indicate (Art. 47) that the place of meeting is at P′, *on*

the left of A and B. Indeed, where m is less than n, or the advance courier is traveling faster than the other, it is evident that they can not meet *in the future.* We may, however, suppose that they *have met before.* We may, therefore, on the principles explained in Art. 164, modify the question in one of two ways.

1st. We may inquire, Where *have* the couriers met? or,

2d. We may suppose the *direction changed*, and call A the advance courier; that is, that they travel toward P′. We shall then have $AB=a$, $AP'=x$, and $BP'=a+x$. Forming and solving the equation as before, we should obtain positive values of x and $a+x$.

3d. If we suppose m equal to n; then,

$$x=\frac{ma}{0}=\infty, \text{ and } x-a=\frac{na}{0}=\infty.$$

This evidently corresponds to the circumstances of the problem; for, if the couriers travel at the same rate, the one can *never* overtake the other. This is sometimes expressed, by saying they only meet at an infinite distance from the point of starting.

4th. If we suppose $a=0$; then,

$$x=\frac{0}{m-n}=0, \text{ and } x-a=\frac{0}{m-n}=0.$$

This corresponds to the circumstances of the problem; for, if the couriers are *no* distance apart, they will have to travel *no* (0) distance to be together.

5th. If we suppose $m=n$, and $a=0$; then, $x=\frac{0}{0}$, and $x-a=\frac{0}{0}$.

But this is the symbol of *indetermination*, and indicates (Art. 137) that the unknown quantity may have *any finite value* whatever. This, also, evidently corresponds to the circumstances of the problem; for, if the couriers are *no* distance apart, and travel at the *same* rate, they will be always together; that is, at *any distance whatever* from the point of starting.

6th. If we suppose $n=0$; then, $x=\frac{am}{m}=a$; that is, the first courier travels from A to B, overtaking the second at B. So, if $m=0$, $x-a=-a$.

7th. If we suppose their rate of travel has a given ratio, as $n=\frac{m}{2}$; then, $x=\frac{2ma}{m}=2a$; that is, the first travels twice the distance from A to B before overtaking the second. The results in the last two cases evidently correspond to the circumstances of the problem.

IV. CASES OF INDETERMINATION IN SIMPLE EQUATIONS AND IMPOSSIBLE PROBLEMS.

167. An **Independent Equation** is one in which the relation of the quantities which it contains can not be obtained directly from others with which it is compared.

Thus,
$$x+3y=19,$$
$$2x+5y=33,$$
are equations which are independent of each other, since the one can not be obtained from the other in a direct manner.

$$x+3y=19,$$
$$2x+6y=38,$$
are not independent of each other, the second being derived directly from the first, by multiplying both sides by 2.

168. An **Indeterminate Equation** is one that can be verified by different values of the same unknown quantity.

Thus, if we have, . $x-y=3$,

By transposition, $x=3+y$.

If we make $y=1$; then, $x=4$. If we make $y=2$; then, $x=5$, and so on; from which it is evident that an unlimited number of values may be given to x and y, that will verify the equation.

If we have two equations containing three unknown quantities, we may eliminate one of them; this will leave one equation containing two unknown quantities, which, as in the preceding example, will be indeterminate.

Thus, in . . . $x+3y-5z=20$,
$$x-y+3z=16,$$

If we eliminate x, we have, after reducing,
$$y-2z=1; \text{ whence, } y=1+2z.$$

If we make $z=1$; then, $y=3$, and $x=20+5z-3y=16$. If we make $z=2$; then, $y=5$, and $x=15$.

So, any number of values of the three unknown quantities may be found that will verify both equations. These examples are sufficient to establish the following

General Principle.— *When the number of unknown quantities exceeds the number of independent equations, the problem is indeterminate.*

A question that involves only one unknown quantity is sometimes indeterminate; the equation deduced from the conditions being identical. (Art. **145**.) The following is an example:

What number is that, whose $\frac{1}{4}$ increased by the $\frac{1}{6}$ is equal to the $\frac{11}{20}$ diminished by the $\frac{2}{15}$?

Let $x=$ the number.

Then, $\frac{x}{4}+\frac{x}{6}=\frac{11x}{20}-\frac{2x}{15}$.

Clearing of fractions, $15x+10x=33x-8x$; or, $25x=25x$; which will be verified by *any value whatever* of x.

169. The reverse of the preceding case requires to be considered; that is, when the number of equations is greater than the number of unknown quantities.

Thus, we may have $2x+3y=23$ (1.)

$3x-2y=2$ (2.)

$5x+4y=40$ (3.)

Each of these equations being independent of the other two, one of them is unnecessary, since the values of x and y, may be found from either two of them.

When a problem contains more conditions than are necessary for determining the values of the unknown quantities, those that are unnecessary are termed *redundant.*

The number of equations may exceed the number of

unknown quantities, so that the values of the unknown quantities shall be incompatible with each other.

Thus, if we have

$$x+\ y=12 \qquad (1.)$$
$$2x+\ y=17 \qquad (2.)$$
$$3x+2y=30 \qquad (3.)$$

The values of x and y, found from equations (1) and (2), are $x=5$, $y=7$; from (1) and (3), $x=6$, and $y=6$; and from (2) and (3), $x=4$, and $y=9$. It is manifest that only two of these equations can be true at the same time.

EXAMPLES TO ILLUSTRATE THE PRECEDING PRINCIPLES.

1. What number is that, which being divided successively by m and n, and the first quotient subtracted from the second, the remainder shall be q? Ans. $x=\frac{mnq}{m-n}$.

What supposition will give a negative solution? Will any supposition give an infinite solution? An indeterminate solution? Illustrate by numbers.

2. Two boats, A and B, set out at the same time, one from C to L, and the other from L to C; the boat A runs m miles, and the boat B, n miles per hour. Where will they meet, supposing it to be a miles from C to L?

Ans. $\frac{ma}{m+n}$ mi. from C, or $\frac{na}{m+n}$ mi. from L.

Under what circumstances will the boats meet half way between C and L? Under what will they meet at C? At L? Above C? Below L? Under what circumstances will they never meet? Under what will they sail together? Illustrate by numbers.

3. Given $2x-y=2$, $5x-3y=3$, $3x+2y=17$, $4x+3y=24$; to find x and y, and show how many equations are redundant. (Art. 169.) Ans. $x=3$, $y=4$.

4. Given $x+2y=11$, $2x-y=7$, $3x-y=17$, $x+3y=19$; to show that the equations are incompatible. (Art. 169.)

V. A SIMPLE EQUATION HAS BUT ONE ROOT.

170. Any simple equation involving only one unknown quantity, (x), may be reduced to the form $mx=n$; for all the terms containing x may be reduced to one term, and all the known quantities to one term; whence, $x=\frac{n}{m}$.

Now, since n divided by m can give but one quotient, we infer that *a simple equation has but one root;* that is, there is but *one* value that will verify the equation.

VI. EXAMPLES INVOLVING THE SECOND POWER OF THE UNKNOWN QUANTITY.

171. It sometimes happens in the solution of an equation, that the second or some higher power of the unknown quantity occurs, but in such a manner that it is easily removed.

The following equations and problems belong to this class:

1. $(4+x)(x-5)=(x-2)^2$.

Performing the operations indicated, we have

$$x^2-x-20=x^2-4x+4.$$

Omitting x^2 on each side, and transposing, we have

$$3x=24, \text{ or } x=8.$$

2. $\frac{(2x+3)x}{2x+1}+\frac{1}{3x}=x+1$. . . . . Ans. $x=1$.

3. $\frac{3x^2-2x+1}{5}=\frac{(7x-2)(3x-6)}{35}+\frac{9}{10}$. Ans. $x=1\frac{5}{68}$.

4. $\frac{4x+3}{6x-43}(3x-19)=2x+19$. . . . . Ans. $x=8$.

5. $\frac{a(b^2+x^2)}{bx}=ac+\frac{ax}{b}$. . . . . Ans. $x=\frac{b}{c}$.

6. $\frac{cx^m}{a+bx}=\frac{dx^m}{e+fx}$. . . . . Ans. $x=\frac{ad-ce}{cf-bd}$.

7. It is required to find a number which being divided into 2 and into 3 equal parts, 4 times the product of the 2 equal parts shall be equal to the continued product of the 3 equal parts. Ans. 27.

8. A rectangular floor is of a certain size. If it were 5 feet broader and 4 feet longer, it would contain 116 feet more; but if it were 4 feet broader and 5 feet longer, it would contain 113 feet more. Required its length and breadth. Ans. Length, 12 feet; breadth, 9 feet.

VI. OF POWERS, ROOTS, RADICALS, AND INEQUALITIES.

I. INVOLUTION, OR FORMATION OF POWERS.

172. The **Power** of a number is the product obtained by multiplying it a certain number of times by itself.

Any number is the *first power* of itself.

When the number is taken *twice* as a factor, the product is called the *second power* or *square* of the number.

When the number is taken *three times* as a factor, the product is called the *third power* or *cube* of the number.

In like manner, the *fourth*, *fifth*, etc., *powers* of a number are the products arising from taking the number, as a factor, *four* times, *five* times, etc.

The **Index** or **Exponent** of the power is the number which denotes the power. It is written to the right of the number, and a little above it.

Thus,	$3=3^1=\ 3$,	is	the	1st	power of	3.
	$3\times3=3^2=\ 9$,	"	"	2d	"	" 3.
	$3\times3\times3=3^3=27$,	"	"	3d	"	" 3.
	$3\times3\times3\times3=3^4=81$,	"	"	4th	"	" 3.
	$\frac{3}{5}\times\frac{3}{5}\times\frac{3}{5}\times\frac{3}{5}=(\frac{3}{5})^4=\frac{81}{625}$,	"	"	4th	"	" $\frac{3}{5}$.
	$a\times a\times a\times a=(a)^4=a^4$	"	"	4th	"	" a.
	$ac^2\times ac^2\times ac^2=(ac^2)^3=a^3c^6$	"	"	3d	"	ac^2.

From the above, we have the following

General Rule for Raising any Quantity to any Required Power.—*Multiply the given quantity by itself, until it is taken as a factor as many times as there are units in the exponent of the required power.*

As the application of this *general rule* frequently involves a tedious operation, it is best to reduce the labor attending it. It will, therefore, be most convenient to divide the subject into distinct cases.

Case I.—To raise a Monomial Quantity to any Power.

By inspecting the illustration above given, it will be seen that a coëfficient is involved by repeated multiplications, as in arithmetic, and the literal factors by repeated additions of the exponents.

Thus, the 3d power of 3 is $3\times3\times3=27$, but the 3d power of a^2 is $a^2\times a^2\times a^2=a^{2+2+2}=a^{2\times3}=a^6$.

If the quantity to be involved is positive, any power of that quantity will be positive. If it is negative, the even powers will be positive and the odd powers negative.

Thus, $-a\times-a=+a^2$, and $-a\times-a\times-a=-a^3$. The 4th power of $-a$ is $+a^4$; the 5th power is $-a^5$; and so on. Hence, we have the following

Rule for Involving a Monomial.—1. *Involve the coëfficient by the rule of arithmetic.*

2. *Multiply the exponents of the literal factors by the exponent of the required power.*

3. *If the quantity be negative, make the even powers positive and the odd powers negative.*

1. Find the square of $5ax^2z^3$. . . . . . Ans. $25a^2x^4z^6$.
2. The square of $-3b^2cd$. . . . . . Ans. $9b^4c^2d^2$.
3. The cube of $2x^2z$. . . . . Ans. $8x^6z^3$.
4. The cube of $-3a^3c^2$. . . . . . . Ans. $-27a^9c^6$.
5. The fourth power of $-2xz^2$. . . . Ans. $16x^4z^8$.
6. The fifth power of $-3a^2b^3$. . . Ans. $-243a^{10}b^{15}$.
7. The seventh power of $-m^2n$. . . Ans. $-m^{14}n^7$.
8. The square of a^mb^{2n}. . . . . Ans. $a^{2m}b^{4n}$.
9. The nth power of xy^2z^p. . . . . Ans. $x^ny^{2n}z^{np}$.
10. The square and the cube of $\frac{2}{5}a^3x^{m+2}y^{p-1}$.
    (1) Ans. $\frac{4}{25}a^6x^{2m+4}y^{2p-2}$. (2) $\frac{8}{125}a^9x^{3m+6}y^{3p-3}$.
11. The square of $\dfrac{ax}{bz^2}$. . . . . Ans. $\dfrac{a^2x^2}{b^2z^4}$.
12. The cube of $\dfrac{2a^2}{3c^3}$. . . . . Ans. $\dfrac{8a^6}{27c^9}$.

Case II.—To square a Binomial Quantity.

The rule for this has already been given, Arts. 78 and 79.

1. Find the square $a-x$. . . . Ans. $a^2-2ax+x^2$.
2. The square of $x+y$. . . . Ans. $x^2+2xy+y^2$.
3. The square of $mx-nx^2$. Ans. $m^2x^2-2mnx^3+n^2x^4$.
4. The square of $\frac{2}{5}a+\frac{1}{2}b$. . . . Ans. $\frac{4}{25}a^2+\frac{2}{5}ab+\frac{1}{4}b^2$.
5. The square of $\dfrac{x+5y}{m^2-n^2}$. . . Ans. $\dfrac{x^2+10xy+25y^2}{m^4-2m^2n^2+n^4}$.

A quantity, consisting of three or four terms, may be squared on the same principle, by reducing it to the form of a binomial, squaring, and completing the operations indicated.

Thus, $a+b-c=a+(b-c)$. Squaring, we have $a^2+2a(b-c)+(b-c)^2=a^2+2ab-2ac+b^2-2bc+c^2$.

$a+b-c+d=(a+b)-(c-d)$. Squaring, $(a+b)^2-2(a+b)(c-d)+(c-d)^2=a^2+2ab+b^2-2ac-2bc+2ad+2bd+c^2-2cd+d^2$.

6. Find the square of $x-\frac{1}{x}-1$. A. $x^2-2x+\frac{1}{x^2}+\frac{2}{x}-1$.

Case III.—To raise a Binomial to the Third Power.

By trial, we find the cube of $a+b$ to be $a^3+3a^2b+3ab^2+b^3$. Hence,

The cube of a binomial is equal to the cube of the first term, plus three times the square of the first into the second, plus three times the first into the square of the second, plus the cube of the second.

If the quantity is a *residual*, as $a-b$, the result will be the same, except that the signs will be alternately *plus* and *minus*. A quantity consisting of three or four terms may be cubed in the same manner, by reducing it to the form of a binomial, as explained above in Case II.

Thus, $(a-b+c)^3=[(a-b)+c]^3=(a-b)^3+3(a-b)^2c+3(a-b)c^2+c^3$, which last may be further expanded.

$(a+b-c+d)^3=[(a+b)-(c-d)]^3=(a+b)^3-3(a+b)^2(c-d)+$, etc.

1. Find the cube of $x+y$. Ans. $x^3+3x^2y+3xy^2+y^3$.
2. The cube of $2x-z$. Ans. $8x^3-12x^2z+6xz^2-z^3$.
3. The cube of $3x+2y$. Ans. $27x^3+54x^2y+36xy^2+8y^3$.
4. The cube of $\frac{m-n}{m-2n}$. Ans. $\frac{m^3-3m^2n+3mn^2-n^3}{m^3-6m^2n+12mn^2-8n^3}$.
5. The cube of $\frac{1}{2}a-\frac{2}{3}b$. Ans. $\frac{1}{8}a^3-\frac{1}{2}a^2b+\frac{2}{3}ab^2-\frac{8}{27}b^3$.
6. Involve $(x-\frac{1}{x})^3$. A. $x^3-3x+\frac{3}{x}-\frac{1}{x^3}=x^3-\frac{1}{x^3}-3(x-\frac{1}{x})$.
7. Involve $(e^x+e^{-x})^3$.
Ans. $e^{3x}+3e^x+3e^{-x}+e^{-3x}=e^{3x}+e^{-3x}+3(e^x+e^{-x})$.
8. Involve $(x+y+z)^3$.
A. $x^3+3x^2y+3x^2z+3xy^2+6xyz+3xz^2+y^3+3y^2z+3yz^2+z^3$.

Case IV.—To raise a Binomial to any Power.

Rules for raising a binomial, or residual quantity, to the 4th, 5th, 6th, or to any higher power, may be formed on the same principle as those given under Case II (See Theorems I and II, Art. 78) and Case III. An easier method, however, was discovered by Sir Isaac Newton, which we now proceed to explain.

NEWTON'S THEOREM.

Let $a+b$ be raised to the sixth power by actual multiplication.

$$\begin{array}{l}
a+b \\
a+b \\
\hline
a^2+ab \\
\quad +ab+b^2 \\
\hline
a^2+2ab+b^2 \quad \text{. . . . . . . second power of } a+b\text{, or } (a+b)^2. \\
a+b \\
\hline
a^3+2a^2b+ab^2 \\
\quad +a^2b+2ab^2+b^3 \\
\hline
a^3+3a^2b+3ab^2+b^3 \quad \text{. . . third power of } a+b\text{, or } (a+b)^3. \\
a+b \\
\hline
a^4+3a^3b+3a^2b^2+ab^3 \\
\quad +a^3b+3a^2b^2+3ab^3+b^4 \\
\hline
a^4+4a^3b+6a^2b^2+4ab^3+b^4 \quad \text{. . . . fourth power, or } (a+b)^4. \\
a+b \\
\hline
a^5+4a^4b+6a^3b^2+4a^2b^3+ab^4 \\
\quad +a^4b+4a^3b^2+6a^2b^3+4ab^4+b^5 \\
\hline
a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5 \quad \text{. . . . . . . } (a+b)^5. \\
a+b \\
\hline
a^6+5a^5b+10a^4b^2+10a^3b^3+5a^2b^4+ab^5 \\
\quad +a^5b+5a^4b^2+10a^3b^3+10a^2b^4+5ab^5+b^6 \\
\hline
a^6+6a^5b+15a^4b^2+20a^3b^3+15a^2b^4+6ab^5+b^6 \quad \text{. . . . } (a+b)^6.
\end{array}$$

If we involve $a-b$, the result will be the same, except that *the signs of the terms will be alternately plus and minus.*

The above results exhibit certain uniform laws of development, following which we may raise a binomial to any required power without the tedious process of multiplication. These laws are as follows:

1st. **Number of Terms.**—*The number of terms in any power of a binomial is one more than the exponent of the power.*

Thus, the 2d power has 3 terms, the 3d power 4 terms, etc.

2d. **Signs of Terms.**—*If both terms of the binomial are positive, all the terms will be positive.*

If the second term is negative, the 1*st*, 3*d*, *etc., or the* ODD *terms, will be positive, and the* EVEN *terms negative.*

3d. **Exponents.**—*The exponent of the* LEADING LETTER *is the same, in the first term, as the power to which the quantity is to be raised, and diminishes by unity, in the succeeding terms, disappearing in the last.*

The FOLLOWING LETTER *is not found in the first term, but enters the second with an exponent of one, which exponent increases, by unity, in the succeeding terms, until it equals, in the last term, the exponent of the power.*

Thus, $(a+b)^6=a^6+a^5b+a^4b^2+a^3b^3+a^2b^4+ab^5+b^6$, omitting coëfficients.

4th. **The Coefficients.**—*The coëfficient of the first and last terms is always unity; that of the second term is the same as the exponent of the* LEADING LETTER *in the first term.*

The coëfficient of any other term may be found by the following rule:

Multiply the coëfficient of any term by the exponent of its leading letter, and divide the product by the number, expressing the place of that term in the series for the coëfficient of the succeeding term.

The coëfficients of all terms equally distant from the first and last are equal.

In the application of this theorem, we may first write the literal factors alone, and afterward supply the coëfficients, according to the rules above given, or, we may carry forward both operations at the same time. Thus,

Let it be required to raise $x+y$ to the 7th power, or to expand $(x+y)^7$.

Literal factors, x^7, x^6y, x^5y^2. x^4y^3, x^3y^4, x^2y^5, xy^6, y^7.

The coëfficient of the 1st term is unity; .. 1st term is x^7.

"	"	"	2d	"	"	7	" 2d	"	" $7x^6y$.
"	"	"	3d	"	"	$\frac{7\times6}{2}$	" 3d	"	" $21x^5y^2$.
"	"	"	4th	"	"	$\frac{21\times5}{3}$	" 4th	"	" $35x^4y^3$.

Continuing thus, we have for the complete expansion,

$$x^7+7x^6y+21x^5y^2+35x^4y^3+35x^3y^4+21x^2y^5+7xy^6+y^7.$$

As a second example by the other method,

Let it be required to expand $(a-b)^6$.

The first term will be a^6; the second, $6a^5b$. For the third, multiply 6 by 5, and divide the product by 2, for the coëfficient, and annex the literal factors. This gives $15a^4b^2$. Multiplying 15 by 4 and dividing by 3. we have for the next term $20a^3b^3$.

Continuing this process, we find the next term to be $15a^2b^4$, the next $6ab^5$, and the last b^6. Giving the proper signs, we have

$$a^6-6a^5b+15a^4b^2-20a^3b^3+15a^2b^4-6ab^5+b^6.$$

The following additional facts may be noted, and may serve to render the application of the above principles still more simple:

1st. The sum of the exponents in every term is the same, and is always equal to the power of the binomial.

Thus, in the first of the above examples, the sum of the exponents in every term is 7; in the second their sum is 6.

2d. If the power of the binomial be even, the number of terms will be odd; but if the power be odd, the number

of terms will be even. In the former case, there will be *one middle term*, and in the latter *two*, to the left and right of which the coëfficients are the same.

Thus, in the above examples, the coëfficients are—

For the 6th power, 1, 6, 15, 20, 15, 6, 1.
For the 7th power, 1, 7, 21, 35, 35, 21, 7, 1.

3d. The sum of the coëfficients, in every case, is equal to 2 raised to the required power of the binomial.

Thus, in the above examples, $1+6+15+20+15+6+1=64=2^6$, and $1+7+21+35+35+21+7+1=128=2^7$.

Expand $(a+b)^4$. . . . Ans. $a^4+4a^3b+6a^2b^2+4ab^3+b^4$.
Expand $(x+y)^6$.
Ans. $x^6+6x^5y+15x^4y^2+20x^3y^3+15x^2y^4+6xy^5+y^6$.
Expand $(a-x)^5$ A. $a^5-5a^4x+10a^3x^2-10a^2x^3+5ax^4-x^5$.
Expand $(a+x)^8$.
Ans. $a^8+8a^7x+28a^6x^2+56a^5x^3+70a^4x^4+56a^3x^5+28a^2x^6+8ax^7+x^8$.
Expand $(a-b)^9$.
Ans. $a^9-9a^8b+36a^7b^2-84a^6b^3+126a^5b^4-126a^4b^5+84a^3b^6-36a^2b^7+9ab^8-b^9$.

If one or both of the terms of the binomial have a coëfficient or exponent greater than unity, or more than one literal factor, the expansion may be made in the same way, after which the operations indicated must be completed.

Thus, $(2x^3+5a^2)^4=(2x^3)^4+4(2x^3)^3(5a^2)+6(2x^3)^2(5a^2)^2+4(2x^3)(5a^2)^3+(5a^2)^4=16x^{12}+160x^9a^2+600x^6a^4+1000x^3a^6+625a^8$.

Or, put $m=2x^3$ and $n=5a^2$. Then, $(2x^3+5a^2)^4=(m+n)^4$. Then,

$$(m+n)^4=m^4+4m^3n+6m^2n^2+4mn^3+n^4.$$

Returning to the values of m and n, we have $m^4=(2x^3)^4=16x^{12}$,

$$4m^3n=4\times(2x^3)^3\times 5a^2=4\times 8x^9\times 5a^2=160x^9a^2.$$
$$6m^2n^2=6\times(2x^3)^2\times(5a^2)^2=6\times 4x^6\times 25a^4=600x^6a^4.$$
$$4mn^3=4\times 2x^3\times(5a^2)^3=4\times 2x^3\times 125a^6=1000x^3a^6.$$
$$n^4=(5a^2)^4=625a^8.$$

Hence, $(2x^3+5a^2)^4=16x^{12}+160x^9a^2+600x^6a^4+1000x^3a^6+625a^8$.

In a similar manner, a quantity consisting of three or four terms may be involved, by first reducing it to the form of a binomial, as explained in Cases II and III.

1. Raise x^2+3y^2 to the fifth power, or expand $(x^2+3y^2)^5$.
   Ans. $x^{10}+15x^8y^2+90x^6y^4+270x^4y^6+405x^2y^8+243y^{10}$.
2. Expand $(2a^2+ax)^3$. Ans. $8a^6+12a^5x+6a^4x^2+a^3x^3$.
3. Expand $(2a+3x)^4$.
   Ans. $16a^4+96a^3x+216a^2x^2+216ax^3+81x^4$.
4. Expand $(\frac{1}{2}a-3b)^4$.
   Ans. $\frac{1}{16}a^4-\frac{3}{2}a^3b+\frac{27}{2}a^2b^2-54ab^3+81b^4$.
5. Cube $a+2b-c$.
   Ans. $a^3+6a^2b-3a^2c+8b^3+12ab^2-12b^2c-c^3+3ac^2$ $+6bc^2-12abc$.
6. Expand $(a+b+c-d)^4$.
   Ans. $a^4+4a^3b+6a^2b^2+4ab^3+b^4+4a^3c+12a^2bc$ $+12ab^2c+4b^3c-4a^3d-12a^2bd-12ab^2d-4b^3d$ $+6a^2c^2+12abc^2+6b^2c^2-12a^2cd-24abcd-12b^2cd$ $+6a^2d^2+12abd^2+6b^2d^2+4ac^3-12ac^2d+12acd^2$ $-4ad^3+4bc^3-12bc^2d+12bcd^2-4bd^3+c^4-4c^3d$ $+6c^2d^2-4cd^3+d^4$.

In many cases, as in some of the examples above given, it will sometimes be found most convenient to involve, by repeated multiplications, under the *general rule*.

For further exercise, take the following:

1. If $x+\frac{1}{x}=p$, show that $x^3+\frac{1}{x^3}=p^3-3p$.

2. If two numbers differ by unity, prove that the difference of their squares is equal to the sum of the numbers.

3. Show that the sum of the cubes of any three consecutive integral numbers is divisible by the sum of those numbers.

NOTE.—For a more general discussion of the Binomial Theorem, see Art. 310.

II. EXTRACTION OF THE SQUARE ROOT.

EXTRACTION OF THE SQUARE ROOT OF NUMBERS.

173. The **Root** of a number is a factor which multiplied by itself a certain number of times will produce the given number.

The **Second** or **Square Root** of a number, is that number which multiplied by itself; that is, taken *twice* as a factor, will produce the given number.

The **Extraction of the Square Root** is the process of finding the second root of a given number.

174. To show the relation that exists between the number of figures in any given number, and the number of figures in its square root, take the first ten numbers and their squares:

1,	2,	3,	4,	5,	6,	7,	8,	9,	10;
1,	4,	9,	16.	25,	36,	49,	64,	81,	100.

The numbers in the first line are also the square roots of the numbers in the second.

We see from this, that the square root of 1 is 1, and the square root of any number less than 100 is either one figure, or one figure and a fraction. Hence,

When the number of places of figures in a number is not more than TWO, *the number of places of figures in the square root will be* ONE.

The square root of 100 is 10; and of any number greater than 100 and less than 10000, the. square root will be less than 100; that is, it will consist of *two* places of figures. Hence,

When the number of places of figures is more than TWO, *and not more than* FOUR, *the number of places of figures in the square root will be* TWO.

In the same manner it may be shown, that when the number of places of figures is more than *four*, and not more than *six*, the number of places in the square root will be *three*, and so on.

175. Every number may be regarded as being composed of tens and units.

Thus, 76 consists of 7 tens and 6 units; and 576 consists of 57 tens and 6 units. Therefore, if we represent the tens by t, and the units by u, any number will be represented by $t+u$, and its square by the square of $t+u$, or $(t+u)^2$.

$$(t+u)^2=t^2+2tu+u^2=t^2+(2t+u)u. \quad \text{Hence,}$$

The square of any number is composed of two quantities, one of which is the square of the tens, and the other twice the tens plus the units multiplied by the units.

Thus, the square of 25, which is equal to 2 tens and 5 units, is

$$\begin{array}{rr} \text{2 tens squared} = (20)^2 = & 400 \\ \text{(4 tens + 5 units) multiplied by } 5 = (40+5)5 = & 225 \\ \hline & 625 \end{array}$$

1. Required to extract the square root of 625.

$$\begin{array}{r|l} \dot{6}2\dot{5} & 25 \\ 400 & \\ \hline 20\times2=40 \quad 225 & \\ 5 \quad 225 & \\ \hline 45 & \end{array}$$

Since the number consists of three places of figures, its root will consist of two places, according to the principle established in Art. 174, we, therefore, separate it into two periods, as in the margin.

Since the square of 2 tens is 400, and of 3 tens is 900, it is evident that the greatest square contained in 600 is the square of 2 tens (20); the square of 2 tens (20) is 400. Subtracting this from 625, the remainder is 225.

The remainder, 225, consists of twice the tens plus the units, multiplied by the units; that is, by the formula, it is $(2t+u)u$, of which t is already found to be 2, and it remains to find u.

Now, the product of the tens by the units can not give a product less than tens; therefore, the unit's figure (5) forms no part of the double product of the tens by the units. Hence, if we divide the remaining figures (22) by the double of the tens (4), the quotient will be the unit's figure, or a figure greater than it.

Dividing 22 by 4 ($2t$) gives 5 (u) for a quotient. This unit's figure (5) is to be added to the double of the tens (40), and the sum multiplied by the unit's figure.

Multiplying $40+5=45(2t+u)$, by 5 (u), the product is 225, which is double the tens plus the units, multiplied by the units. As there is no remainder, we conclude that 25 is the exact square root of 625.

In squaring and doubling the tens, it is customary to omit the ciphers, and to add the unit's figure to the double of the tens, by merely writing it in the unit's place. The actual operation is usually performed as in the margin.

```
   625|25
   400
45|225
  |225
```

2. Required to extract the square root of 59049.

```
     59049|243
     4
 44|190
   |176
483|1449
   |1449
```

Since this number consists of five places of figures, its square root will consist of three places. (Art. 174.) We, therefore, separate it into *three* periods.

In performing this operation, we find the square root of the number 590, on the same principle as in the preceding example.

We next consider 24 as so many tens, and proceed to find the unit's figure (3) as in the preceding example.

From these illustrations, we derive the following

Rule for the Extraction of the Square Root of Numbers.—1st. *Separate the given number into periods of two places each, beginning at the unit's place.* (The left period will often contain but one figure.)

2d. *Find the greatest square in the left period, and place its root on the right, after the manner of a quotient in division. Subtract the square of the root from the left period, and to the remainder bring down the next period for a dividend.*

3d. *Double the root already found, and place it on the left for a divisor. Find how many times the divisor is contained in the dividend, exclusive of the right hand figure, and place the figure in the root and also on the right of the divisor.*

4th. *Multiply the divisor thus increased by the last figure of the root; subtract the product from the dividend, and to the remainder bring down the next period for a new dividend.*

5th. *Double the whole root already found for a new divisor, and continue the operation as before, until all the periods are brought down.*

NOTE.—If, in any case, the division can not be effected, place a cipher in the root and divisor, and bring down the next period.

176. In extracting the square root of numbers, the remainder may sometimes be greater than the divisor, while the last figure of the root can not be increased. To explain this,

Let a and $a+1$. be two consecutive numbers.

Then, $(a+1)^2=a^2+2a+1$, the square of the greater.
And $(a)^2=a^2$, " " " less.

Their difference is $2a+1$. Hence,

The difference of the squares of two consecutive numbers is equal to twice the less number, increased by unity.

Therefore, when any remainder is less than twice the root already found, plus one, the last figure can not be increased.

Required the square root of

1. 2601. . . . Ans. 51.	5. 43046721. Ans. 6561.
2. 7225. . . . Ans. 85.	6. 49042009. Ans. 7003.
3. 47089. . . Ans. 217.	7. 1061326084. A. 32578.
4. 390625. . Ans. 625.	8. 943042681. Ans. 30709.

EXTRACTION OF THE SQUARE ROOT OF FRACTIONS.

177. Since $\frac{2}{5}\times\frac{2}{5}=\frac{4}{25}$, the square root of $\frac{4}{25}$ is $\frac{2}{5}$; that is, $\sqrt{\frac{4}{25}}=\frac{\sqrt{4}}{\sqrt{25}}=\frac{2}{5}$. Hence, we have the following

Rule for Extracting the Square Root of a Fraction.—*Extract the square root of both terms.*

When the terms of a fraction are not perfect squares, they may sometimes be made so by reducing. Thus,

Find the square root of $\frac{20}{45}$.

Here, $\frac{20}{45}=\frac{4\times5}{9\times5}$. By canceling the common factor 5, the fraction becomes $\frac{4}{9}$, the square root of which is $\frac{2}{3}$.

When both terms are perfect squares, and contain a common factor, the reduction may be made either before or after the square root is extracted.

Thus, $\sqrt{\frac{36}{81}}=\frac{6}{9}=\frac{2}{3}$; or, $\frac{36}{81}=\frac{4}{9}$, and $\sqrt{\frac{4}{9}}=\frac{2}{3}$.

Required the square root of

1. $\frac{64}{121}$. . . . Ans. $\frac{8}{11}$.
2. $\frac{225}{400}$. . . . Ans. $\frac{3}{4}$.
3. $\frac{9747}{10092}$. . . Ans. $\frac{57}{58}$.
4. $\frac{56169}{1000000}$. Ans. $\frac{237}{1000}$.

178. A **Perfect Square** is a number whose square root can be exactly ascertained; as, 4, 9, 16, etc.

An **Imperfect Square** is a number whose square root can not be exactly ascertained; as, 2, 3, 5, 6, etc.

Since the difference of two consecutive square numbers, a^2 and a^2+2a+1, is $2a+1$; therefore, there are always $2a$ imperfect squares between them.

Thus, between the square of 5 (25) and the square of 6 (36), there are 10 ($2a=2\times5$) imperfect squares.

A quantity, affected by a radical sign, whose root can not be exactly found, is called a *radical*, or *surd*, or *irrational root;* as, $\sqrt{2}$, $\sqrt[3]{5}$, etc.

The root of such a quantity, expressed with more or less accuracy in decimals, is called the *approximate value*, or *approximate root.* Thus, 1.414+ is the approximate value of $\sqrt{2}$.

179. It might be supposed, that when the square root of a whole number can not be expressed by a whole number, it might be exactly equal to some fraction. That it can not, will now be shown.

Let c be an imperfect square, as 2, and, *if possible*, let its square root be a fraction, $\frac{a}{b}$, in its *lowest* terms.

Then, $\sqrt{c}=\frac{a}{b}$; and $c=\frac{a^2}{b^2}$, by squaring both sides (Art. 148).

Now, by supposition, a and b have no common factor; therefore, their squares, a^2 and b^2, can have no common factor, since to square a number, we merely repeat its factors. Consequently, $\frac{a^2}{b^2}$ must be in its lowest terms, and can not be equal to a whole number. Hence, the equation $c=\frac{a^2}{b^2}$, is not true, and the *supposition* on which it is founded, that is, that $\sqrt{c}=\frac{a}{b}$, is *false;* therefore, the square root of an imperfect square can not be a fraction.

APPROXIMATE SQUARE ROOTS.

180. To explain the method of finding the approximate square root of an imperfect square, let it be required to find the square root of 5 to within $\frac{1}{3}$.

If we reduce 5 to a fraction whose denominator is 9 (the square of 3, the denominator of the fraction $\frac{1}{3}$), we have $5=\frac{45}{9}$.

Now, the square root of $\frac{45}{9}$ is greater than $\frac{6}{3}$, and less than $\frac{7}{3}$; hence, $\frac{6}{3}$, or 2, is the square root of 5 to within $\frac{1}{3}$.

To generalize this explanation, let it be required to extract the square root of a to within a fraction $\frac{1}{n}$.

Write a (Art. 127) under the form $\frac{an^2}{n^2}$, and denote the entire part of the square root of an^2 by r. Then, an^2 will be comprised between r^2 and $(r+1)^2$, and the square root of $\frac{an^2}{n^2}$ will be comprised between $\frac{r}{n}$ and $\frac{r+1}{n}$.

But the difference between $\frac{r}{n}$ and $\frac{r+1}{n}$ is $\frac{1}{n}$; therefore, $\frac{r}{n}$ represents the square root of a to within $\frac{1}{n}$. Hence,

Rule for Extracting the Square Root of a Whole Number to within a Given Fraction.—1. *Multiply the given number by the square of the denominator of the fraction, which determines the degree of approximation.*

2. *Extract the square root of this product to the nearest unit, and divide the result by the denominator of the fraction.*

1. Find the square root of 3 to within $\frac{1}{3}$. . Ans. $1\frac{2}{3}$.
2. Of 10 to within $\frac{1}{4}$. . . . . Ans. 3.
3. Of 19 to within $\frac{1}{6}$. . . . . Ans. $4\frac{1}{3}$.
4. Of 30 to within $\frac{1}{10}$. . . . . Ans. 5.4.
5. Of 75 to within $\frac{1}{100}$. . . . . Ans. 8.66.

Since the square of 10 is 100, the square of 100, 10000, and so on, the number of ciphers in the denominator of a decimal fraction is *doubled* by squaring it. Therefore,

When the fraction which determines the degree of approximation is a decimal, add two ciphers for each decimal place required; and, after extracting the square root, point off from the right one place of decimals for each two ciphers added.

6. Find the square root of 3 to five places of decimals. Ans. 1.73205.
7. Find the square root of 7 to five places of decimals. Ans. 2.64575.
8. Find the square root of 500. Ans. 22.360679+.

181. To find the approximate square root of a fraction.

1. Required to find the square root of $\frac{4}{7}$ to within $\frac{1}{7}$.

$$\tfrac{4}{7}=\tfrac{4}{7}\times\tfrac{7}{7}=\tfrac{28}{49}.$$

The square root $\frac{28}{49}$ is greater than $\frac{5}{7}$ and less than $\frac{6}{7}$; therefore, $\frac{5}{7}$ is the square root of $\frac{4}{7}$ to within less than $\frac{1}{7}$. Hence, to find the square root of a fraction to within one of its equal parts,

Rule.—*Multiply the numerator by its denominator, extract the square root of the product to the nearest unit, and divide the result by the denominator.*

2. Find the square root of $\frac{7}{11}$ to within $\frac{1}{11}$. Ans. $\frac{8}{11}$.

3. Find the square root of $\frac{17}{20}$ to within $\frac{1}{20}$. Ans. $\frac{9}{10}$.

It is obvious that any decimal, or whole number and decimal, may be written in the form of a common fraction, and having its denominator a perfect square, by adding ciphers to both terms. Thus, $.3=\frac{3}{10}=\frac{30}{100}$; $.156=\frac{1560}{10000}$; $1.2=\frac{120}{100}$, and so on.

Therefore, to extract the square root, as in the method for the approximate square root of a whole number (Art. 180),

Rule.—1. *Annex ciphers to the decimal, until the number of decimal places shall be equal to double the number required in the root.*

2. *After extracting the root, point off from the right the required number of decimal places.*

4. Find the square root of .4 to six places.
Ans. .632455+.

5. Find the square root of 7.532 to five places.
Ans. 2.74444+.

When the denominator of a fraction is a perfect square, extract the square root of the numerator to as many places of decimals as are required, and divide the result by the square root of the denominator.

Or, reduce the fraction to a decimal, and then extract its square root. When the denominator of the fraction is not a perfect square, the latter method should be used.

6. Find the square root of $\frac{5}{16}$ to five places.

$$\sqrt{5}=2.23606+,\quad \sqrt{16}=4,\quad \sqrt{\tfrac{5}{16}}=\frac{2.23606+}{4}=.55901+.$$

Or, $\frac{5}{16}=.3125$, and $\sqrt{.3125}=.55901+$.

7. Find the square root of $\frac{3}{5}$. . . . Ans. .774596+.

8. Find the square root of $1\frac{1}{4}$. . . . Ans. 1.11803+.

9. Find the square root of $3\frac{5}{8}$. . . . Ans. 1.903943+.

10. Find the square root of $11\frac{2}{9}$. . . . Ans. 3.349958+.

EXTRACTION OF THE SQUARE ROOT OF ALGEBRAIC QUANTITIES.

EXTRACTION OF THE SQUARE ROOT OF MONOMIALS.

182. To square a monomial, (Art. 172), we square its coëfficient, and multiply each exponent by 2.

Thus, $(3mn^2)^2=9m^2n^4$.
Therefore, $\sqrt{9m^2n^4}=3mn^2$. Hence, we have the following

Rule for Extracting the Square Root of a Monomial.—*Extract the square root of the coëfficient as a number, and divide the exponent of each letter by* 2.

Since $+a\times+a=+a^2$, $-a\times-a=+a^2$;
Therefore, $\sqrt{a^2}=+a$, or $-a$. Hence,

The square root of any positive quantity is either *plus* or *minus*. This is expressed by writing the double sign before the root. Thus, $\sqrt{4a^2}=\pm2a$; read, *plus or minus* $2a$.

If a monomial is *negative*, the extraction of the square root is impossible, since the square of any quantity, either positive or negative, is necessarily positive. Thus, $\sqrt{-4}$, $\sqrt{-b}$, are algebraic symbols, which indicate impossible operations.

Such expressions are termed *imaginary quantities*. In an equation of the second degree, they often indicate some absurdity, or impossibility in the equation or problem from which it was derived.

1. $16x^2y^4$. Ans. $\pm4xy^2$.
2. $25m^2n^2$. Ans. $\pm5mn$.
3. $m^2x^4y^6z^8$. Ans. $\pm mx^2y^3z^4$.
4. $1024a^2b^6z^{10}$. Ans. $\pm32ab^3z^5$.

Since $\left(\frac{a}{b}\right)^2=\frac{a}{b}\times\frac{a}{b}=\frac{a^2}{b^2}$; therefore, $\sqrt{\frac{a^2}{b^2}}=\frac{\sqrt{a^2}}{\sqrt{b^2}}=\pm\frac{a}{b}$. Hence,

To find the square root of a monomial fraction, extract the square root of both terms.

5. Find the square root of $\frac{a^2b^4}{c^2d^6}$. . . . Ans. $\pm\frac{ab^2}{cd^3}$.

6. Find the square root of $\frac{4x^2y^2}{25a^2b^4}$. . Ans. $\pm\frac{2xy}{5ab^2}$.

EXTRACTION OF THE SQUARE ROOT OF POLYNOMIALS.

183. In order to deduce a rule for extracting the square root of polynomials, let us first find the relation that exists between the several terms of any quantity and its square.

$(a+b)^2=a^2+2ab+b^2=a^2+(2a+b)b$.

$(a+b+c)^2=a^2+2ab+b^2+2ac+2bc+c^2=a^2+(2a+b)b+(2a+2b+c)c$.

$(a+b+c+d)^2=a^2+2ab+b^2+2ac+2bc+c^2+2ad+2bd+2cd+d^2=a^2+(2a+b)b+(2a+2b+c)c+(2a+2b+2c+d)d$.

Or, by calling the successive terms of a polynomial, r, r', r'', r''', and so on, we shall have $(r+r'+r''+r''')^2=r^2+(2r+r')r'+(2r+2r'+r'')r''+(2r+2r'+2r''+r''')r'''$, where the law is manifest.

In this formula, r, r', r'', r''', may represent any algebraic quantities whatever, either integral or fractional, positive or negative.

This formula gives the following *law:*

The square of any polynomial is equal to the square of the first term — plus twice the first term, plus the second, multiplied by the second — plus twice the first and second terms, plus the third, multiplied by the third — plus twice the first, second, and third terms, plus the fourth, multiplied by the fourth, and so on.

Hence, by reversing the operation, we have the following

Rule for Extracting the Square Root of a Polynomial.—1st. *Arrange the polynomial with reference to a certain letter.*

2d. *Extract the square root of the first term, place the result on the right, and subtract its square from the given quantity.*

3d. *Divide the first term of the remainder by double the part of the root already found, and annex the result to both the root and the divisor. Multiply the divisor thus increased by the second term of the root, and subtract the product from the remainder.*

4th. *Double the terms of the root already found for a partial divisor, divide the first term of the remainder by the first term of the divisor, and proceed in a similar manner to find the other terms.*

1. Find the square root of $4x^2y^2+12x^2y+9x^2-30xy^2-20xy^3+25y^4$.

Arranging the polynomials with reference to y, we have

$$\begin{array}{r|l|l}
 & 25y^4-20xy^3+4x^2y^2-30xy^2+12x^2y+9x^2 & \underline{5y^2-2xy-3x} \text{ ROOT.} \\
 & 25y^4 & \\
10y^2-2xy & -20xy^3+4x^2y^2 & \\
 & -20xy^3+4x^2y^2 & \\
10y^2-4xy-3x & -30xy^2+12x^2y+9x^2 & \\
 & -30xy^2+12x^2y+9x^2 &
\end{array}$$

If the preceding example be arranged according to the powers of x, the root found will be $3x+2xy-5y^2$. This is correct also, as may be shown generally, thus:

$$\sqrt{(a^2+2ax+x^2)}=\pm(a+x)=a+x, \text{ or } -a-x.$$

2. $x^2+6ax+9a^2$. . . . . Ans. $x+3a$.
3. $16x^2-40xy+25y^2$. . . . . Ans. $4x-5y$.
4. $4x^2z^2-12xyz+9y^2$. . . . . Ans. $2xz-3y$.
5. $49a^{4m-6}-42a^{6m-2}+9a^{8m+2}$. . . Ans. $7a^{2m-3}-3a^{4m+1}$.
6. $1+2x+7x^2+6x^3+9x^4$. . . . . . Ans. $1+x+3x^2$.
7. $9a^4-12a^3b+34a^2b^2-20ab^3+25b^4$.
Ans. $3a^2-2ab+5b^2$.
8. $x^6+4x^5+10x^4+20x^3+25x^2+24x+16$.
Ans. x^3+2x^2+3x+4.
9. $9x^2-6xy+30xz+6xt+y^2-10yz-2yt+25z^2+10zt+t^2$.
Ans. $3x-y+5z+t$.

10. $x^4-2x^3+\frac{3x^2}{2}-\frac{x}{2}+\frac{1}{16}$. . . . . . Ans. $x^2-x+\frac{1}{4}$.

11. $\frac{25a^2b^2}{4}-\frac{5abc^2}{3}+\frac{c^4}{9}$. . . . . Ans. $\frac{5ab}{2}-\frac{c^2}{3}$.

12. $\frac{1051x^2}{25}-\frac{6x}{5}-\frac{14x^3}{5}+9+49x^4$. Ans. $7x^2-\frac{x}{5}+3$.

13. $\frac{a^2}{b^2}-2+\frac{b^2}{a^2}$. . . . . Ans. $\frac{a}{b}-\frac{b}{a}$.

14. Reduce the following expression to its simplest form, and extract the square root:

$(a-b)^4-2(a^2+b^2)(a-b)^2+2(a^4+b^4)$. Ans. a^2+b^2.

15. Find the square root of $1-x^2$ to five terms.

Ans. $1-\frac{x^2}{2}-\frac{x^4}{8}-\frac{x^6}{16}-\frac{5x^8}{128}-$, etc.

16. Find the first five terms of the square root of x^2+a^2.

Ans. $x+\frac{a^2}{2x}-\frac{a^4}{8x^3}+\frac{a^6}{16x^5}-\frac{5a^8}{128x^7}+$, etc.

184. The following remarks will be found useful:

1st. *No binomial can be a perfect square;* for the square of a monomial is a monomial, and the square of a binomial is a trinomial.

Thus, a^2+b^2 is not a perfect square, but if we add to it, or subtract from it, $2ab$, it becomes the square of $a+b$ or of $a-b$.

2d. In order that a *trinomial* may be a perfect square, the two extreme terms must be perfect squares, and the middle term double the product of the square roots of the extreme terms.

Hence, to find the square root of a trinomial when it is a perfect square,

Extract the square roots of the extreme terms, and unite them by the sign of the second term.

Thus, $a^2+4ax+4x^2$ is a perfect square, and its square root is, $a+2x$; $4x^2+8xy+9y^2$ is not a perfect square. For other illustrations, see Exs. 2, 3, 4, 11, and 13, Art. 183.

III. EXTRACTION OF THE CUBE ROOT.

EXTRACTION OF THE CUBE ROOT OF NUMBERS.

185. The **Cube,** or *third* power of a number, is the product arising from taking it *three* times as a factor. (Art. 172.)

The **Cube Root,** or *third root* of a number, is one of three equal factors into which it may be resolved.

To extract the cube root of a number, is to find a number which, taken *three* times as a factor, will produce the given number.

186. To show the relation that exists between the number of figures in any given number, and the number of figures in its cube root.

The first ten numbers and their cubes are,

Roots,	1,	2,	3,	4,	5,	6,	7,	8,	9,	10;
Cubes,	1,	8,	27,	64,	125,	216,	343,	512,	729,	1000.

We see from this that the cube of a number consisting of one place of figures, does not exceed three places.

Again, comparing the numbers 10 and 100, we have,

Numbers, ...	10,	100;
Cubes, ...	$\dot{1}00\dot{0}$,	$\dot{1}00\dot{0}00\dot{0}$.

Since the cube of 10 is $\dot{1}00\dot{0}$, and the cube of 99, which is less than 100, is less than $\dot{1}00\dot{0}00\dot{0}$; therefore, the cube of a number consisting of two places of figures, has more than *three* places, and not more than *six* places of figures.

Again, since the cube of 100 is $\dot{1}00\dot{0}00\dot{0}$, and the cube of 1000 is $\dot{1}00\dot{0}00\dot{0}00\dot{0}$; the cube of a number consisting of three places of figures has more than *six* places, and not more than *nine* places of figures.

If, therefore, we begin at the unit's place of a number, and separate it into periods of three places each, the number of periods will show the number of places of figures in the root. The left period will often contain only one or two figures.

187. To investigate a rule for the extraction of the cube root.

The first step is to show the relation that exists between any number composed of units and tens, and its cube.

Let . . $t=$ the tens, and $u=$ the units of a given number.
Then, $t+u=$ the number.
And $(t+u)^3=$ the cube of the number.
But $(t+u)^3=t^3+3t^2u+3tu^2+u^3=t^3+(3t^2+3tu+u^2)u$. Hence,

The cube of any number consisting of tens and units, is equal to the cube of the tens, — plus three times the square of the tens, plus three times the product of the tens and units, plus the square of the units, all three multiplied by the units.

1. Required to extract the cube root of 13824.

$$\begin{array}{r|l} & tu \\ & 13\dot{8}2\dot{4}\,|\,24 \\ & 8 \\ 3t^2=1200 & 5824 \\ 3tu=240 & \\ u^2=16 & \\ \hline 1456 & 5824 \end{array}$$

Separating the number into periods by points, we find there will be two figures in the root. The greatest cube in 13 (thousand) is 8 (thousand); the cube root of which is 2 (t); and its cube, 8 (thousand), corresponds to t^3 in the formula.

We then subtract this from the given number, and find a remainder 5824, which corresponds to $(3t^2+3tu+u^2)u$ in the formula. The first term, $3t^2$, of this formula, is sometimes termed the trial divisor, as it is used to find the unit's figure u.

If the remaining terms were only $3t^2u$, we could readily find u by dividing by $3t^2$; but if we divide by $3t^2$, we may obtain a figure too large, on account of omitting the terms $3tu+u^2$, of which u is as yet unknown. But if we first obtain a figure too large, at a second trial we must take one that is less.

Since the square of tens is hundreds, in using three times the square of the ten's figure as a trial divisor, we omit the figures (24) in the unit's and ten's places of the dividend.

In this case, 12 is contained in 58 four times. This gives 4 (u) for the required unit's figure, and we now find the complete divisor, $3t^2+3tu+u^2=1200+240+16=1456$.

Multiplying this by 4 (u), the product is 5824, which, subtracted from the first remainder, leaves zero (0), and shows that 24 is the exact cube root required.

In cubing the tens, it is customary to omit the ciphers; but in taking three times the square of the tens, also in taking three times the product of the tens by the units, it is best to write ciphers in the vacant orders.

2. Required to find the cube root of 44361864.

After separating the number into periods, we find the cube root (35) of 44361 on the same principles as in the preceding example. Then, considering 35 ($10h+t$) as so many tens, we find the unit's figure (4), as in the preceding example.

In dividing by the trial divisor 27, to find the second figure (5), we first obtain 6, but this is found by trial to be too large.

$$\begin{array}{r|l}
 & htu \\
 & \dot{4}4\dot{3}6\dot{1}86\dot{4}\,|\,\underline{354} \\
 & 27 \\ \hline
3h^2=2700 & 17361 \\
3ht=\ 450 & \\
t^2=\ \ 25 & \\ \hline
3175 & 15875 \\ \hline
3(h+t)^2=367500 & 1486864 \\
3(h+t)u=\ 4200 & \\
u^2=\ \ 16 & \\ \hline
371716 & \underline{1486864}
\end{array}$$

From the preceding, we derive the following

Rule for the Extraction of the Cube Root of Numbers.—1st. *Separate the given number into periods of three places each, beginning at the unit's place.* (The left period will often contain but one or two figures.)

2d. *Find the greatest cube in the left period, and place its root on the right, as in division. Subtract the cube of the root from the left period, and to the remainder bring down the next period for a dividend.*

3d. *Square the root already found, and multiply it by 3 for a trial divisor. Find how many times this divisor is contained in the dividend, omitting the unit's and ten's figures, and write the result in the root. Add together, the trial divisor with two ciphers annexed; three times the product of the last figure of the root by the rest, with one cipher annexed;*

and the square of the last figure; the sum will be the complete divisor.

4th. *Multiply the complete divisor by the last figure of the root, and subtract the product from the dividend, and to the remainder bring down the next period for a new dividend, and so proceed until all the periods are brought down.*

Extract the cube root of the following numbers:

3. 12167. . . Ans. 23.
4. 39304. . . Ans. 34.
5. 493039.. Ans. 79.
6. 2097152. Ans. 128.
7. 127263527. Ans. 503.
8. 403583419. Ans. 739.
9. 158252632929. Ans. 5409.

By a process of reasoning similar to that given in Art. 177, we deduce the following

Rule for Extracting the Cube Root of a Fraction.—*Reduce the fraction, if necessary, to its lowest terms, and extract the cube root of both terms.*

10. Find the cube root of $\frac{64}{125}$. . . . . Ans. $\frac{4}{5}$.
11. Find the cube root of $\frac{216}{2744}$. . . . Ans. $\frac{3}{7}$.

188. A **Perfect Cube** is a number whose cube root can be exactly ascertained; as, 8, 27, 64, etc.

An **Imperfect Cube** is a number whose cube root can not be exactly ascertained; as, 2, 3, 4, etc.

It may be shown, by a course of reasoning precisely similar to that employed in Art. 179, that *the cube root of an imperfect cube can not be a fraction.*

APPROXIMATE CUBE ROOTS.

189. To illustrate the method of finding the approximate cube root of an imperfect cube, let it be required to find the cube root of 6 to within $\frac{1}{4}$. $6=\frac{384}{64}$.

Now, the cube root of 384 is greater than 7 and less than 8; therefore, the cube root of $\frac{384}{64}$ is greater than $\frac{7}{4}$ and less than $\frac{8}{4}$; hence, $\frac{7}{4}$ is the cube root of 6 to within less than $\frac{1}{4}$.

To generalize this method, let it be required to extract the cube root of a number a, to within a fraction $\frac{1}{n}$.

$$a=\frac{a\times n^3}{1\times n^3}=\frac{an^3}{n^3}.$$

Let r be the root of the greatest cube contained in an^3; then, $\frac{an^3}{n^3}$ is comprised between $\frac{r^3}{n^3}$ and $\frac{(r+1)^3}{n^3}$; hence, its cube root is comprised between $\frac{r}{n}$ and $\frac{r+1}{n}$; and since the difference of these fractions is $\frac{1}{n}$; therefore, $\frac{r}{n}$ is the cube root of a to within $\frac{1}{n}$. Hence,

Rule for Extracting the Cube Root of a Whole Number to within a Given Fraction.—*Multiply the given number by the cube of the denominator of the fraction which determines the degree of approximation; extract the cube root of this product to the nearest unit, and divide the result by the denominator of the fraction.*

2. Find the cube root of 5 to within $\frac{1}{5}$. . . Ans. $1\frac{3}{5}$.
3. Find the cube root of 10 to within $\frac{1}{8}$. . . Ans. $2\frac{1}{8}$.

Since the cube of 10 is 1000, the cube of 100, 1000000, and so on, the number of ciphers in the cube of the denominator of a decimal fraction is equal to three times the number in the denominator itself. Therefore,

When the fraction which determines the degree of approximation is a decimal, add three ciphers for each decimal place required; and after extracting the root, point off from the right one place of decimals for each three ciphers added.

4. Find the cube root of 2 to five places. A. 1.25992.
5. Find the cube root of 37 to six places. A. 3.332222.

By adding ciphers to both terms, any decimal, or whole number and decimal, may be written in the form of a

fraction, having its denominator a perfect cube; thus, $.2=\frac{200}{1000}$, $.25=\frac{250}{1000}$, $6.4=\frac{6400}{1000}$, and so on. Therefore, to find the cube root,

Annex ciphers to the given decimal, until the number of decimal places shall be equal to three times the number required in the root. Extract the root, and point off from the right the required number of decimal places.

6. Find the cube root of .4 to four places. Ans. .7368.
7. Find the cube root of 34.3 to six places.
Ans. 3.249112.

To find the cube root of a fraction or a mixed number, first reduce the fraction to a decimal.

8. Find the cube root of $\frac{5}{9}$. . . . Ans. .82207+.
9. Find the cube root of $5\frac{104}{105}$. . . Ans. 1.816+.
10. Divide the cube root of $\frac{2515.456}{32768}$ by the square root of the square root of 8.3521. Ans. .25.

EXTRACTION OF THE CUBE ROOT OF ALGEBRAIC QUANTITIES.

EXTRACTION OF THE CUBE ROOT OF MONOMIALS.

190. If we cube, for example, $2ax^2$, we have $(2ax^2)^3 = 8a^3x^6$; that is, we cube the coëfficient, and multiply the exponent of each letter by 3. Hence, conversely, we have the following

Rule for Extracting the Cube Root of a Monomial.—*Extract the cube root of the coëfficient, and divide the exponent of each letter by 3.*

Find the cube root of the following Monomials:

1. $8x^3z^6$. . . Ans. $2xz^2$.
2. $27x^6y^{15}$. . . Ans. $3x^2y^5$.
3. $-64a^3m^6$. Ans. $-4am^2$.
4. $a^{3m+3c}x^6$. Ans. $a^{m+c}x^2$.

Since, $\left(\frac{a}{b}\right)^3=\frac{a^3}{b^3}$; therefore, $\sqrt[3]{\frac{a^3}{b^3}}=\frac{a}{b}$. Hence,

To find the cube root of a monomial fraction, extract the cube root of both terms.

5. Find the cube root of $\frac{8a^3}{27x^6}$. . . . . Ans. $\frac{2a}{3x^2}$.

6. Find the cube root of $-\frac{64x^3y^6}{125m^3n^9}$. . Ans. $-\frac{4xy^2}{5mn^3}$.

EXTRACTION OF THE CUBE ROOT OF POLYNOMIALS.

191. To investigate a rule for extracting the cube root of polynomials, let us first examine the relation that exists between a polynomial and its cube.

$(a+b)^3=a^3+3a^2b+3ab^2+b^3=a^3+(3a^2+3ab+b^2)b.$

$(a+b+c)^3=\{(a+b)+c\}^3=(a+b)^3+\{3(a+b)^2+3(a+b)c+c^2\}c.$

$(a+b+c+d)^3=\{(a+b+c)+d\}^3=(a+b+c)^3+\{3(a+b+c)^2+3(a+b+c)d+d^2\}d.$

Hence, the cube of a polynomial is formed according to the following law:

The cube of a polynomial is equal to the cube of the first term — plus three times the square of the first term, plus three times the product of the first term by the second, plus the square of the second, all three multiplied by the second — plus three times the square of the first two terms, plus three times the product of the first two terms by the third, plus the square of the third, all three multiplied by the third, and so on.

By reversing this law, we derive the following

Rule for Extracting the Cube Root of a Polynomial.—
1st. *Arrange the polynomial with reference to a certain letter.*

2d. *Extract the cube root of the first term for the first term of the root, and subtract its cube from the given polynomial.*

3d. *Take three times the square of the first term of the root, and call it a trial divisor for finding each of the remaining terms of the root. Find how often the trial divisor is contained in the first term of the remainder; this will give the second term of the root. Then form a complete divisor by adding together three times the square of the first term of the root, plus three times the product of the first term by the second, plus the square of the second. Multiply these by the second term of the root, and subtract the product from the first remainder.*

4th. *Again, find how often the trial divisor is contained in the first term of the remainder; this will give the third term of the root. Then form a complete divisor as before, by adding together three times the square of the first and second terms, plus three times the product of the first and second terms by the third, plus the square of the third. Multiply these by the third term of the root, and subtract the product from the last remainder.*

5th. *Continue thus till all the terms of the root are found.*

1. Find the cube root of $x^6-6x^5+12x^4+3a^2x^4-8x^3$ $12a^2x^3+12a^2x^2+3a^4x^2-6a^4x+a^6$.

$$x^6-6x^5+12x^4+3a^2x^4-8x^3-12a^2x^3+12a^2x^2+3a^4x^2-6a^4x+a^6 \;\underline{|x^2-2x+a^2}$$

$$x^6$$

$$3x^4-6x^3+4x^2)\quad -6x^5+12x^4-8x^3$$

$$-6x^5+12x^4-8x^3$$

$$3x^4-12x^3+12x^2+3a^2x^2-6a^2x+a^4)+3a^2x^4-12a^2x^3+12a^2x^2$$

$$+3a^4x^2-6a^4x+a^6.$$

To bring the work within the page, the last remainder and subtrahend are each written in two lines.

$$+3a^2x^4-12a^2x^3+12a^2x^2$$

$$+3a^4x^2-6a^4x+a^6.$$

We first extract the cube root of x^6, which gives x^2 for the first term of the required root. Then, 3 times the square of this, $=3x^4$, constitutes the *trial divisor* for finding the remaining terms.

Dividing $-6x^5$ by $3x^4$, gives $-2x$, the second term of the root. We then form the complete divisor by adding together $3(x^2)^2+3(x^2\times-2x)+(-2x)^2=3x^4-6x^3+4x^2$. Multiplying this by $-2x$,

and subtracting, the first term of the second remainder is $+3a^2x^4$, which divided by the trial divisor, gives $+a^2$, for the third term of the root, and so on.

SECOND METHOD.

The following rule, applicable both to numerical and algebraic quantities, may be found more convenient in some cases. The principle upon which it is founded will be obvious upon a careful inspection of the full expansion of the forms $(a+b)^3$, $(a+b+c)^3$, etc.

1. *Arrange the polynomial, as in the previous rule.*
2. *Extract the cube root of the first term, etc., as before.*
3. *Find the trial divisor and 2d term of the root, as before.*
4. *Cube the root already found, and subtract the result from the given polynomial.*
5. *Divide the first term of the remainder by the same trial divisor for the third term of the root. Cube the root already found, and subtract the result from the given polynomial. Continue this process until a quantity is found in the root which will be equal, when cubed, to the given polynomial.*

To illustrate this rule, take the example given above.

$$x^6-6x^5+12x^4+3a^2x^4-8x^3-12a^2x^3+12a^2x^2+3a^4x^2-6a^4x+a^6 \quad \underline{|x^2-2x+a^2}$$

$$x^6$$

$$\underline{3x^4}|-6x^5+12x^4, \text{ etc., 1st remainder.}$$

$$x^6-6x^5+12x^4-8x^3, \text{ cube of } x^2-2x.$$

$$\underline{3x^4}|\quad 3a^2x^4-12a^2x^3, \text{ etc., 2d remainder.}$$

$$x^6-6x^5+12x^4+3a^2x^4-8x^3-12a^2x^3+12a^2x^2+3a^4x^2-6a^4x+a^6$$

We first extract the cube root of x^6, and find it x^2. Cubing this, subtracting, and dividing the first term of the remainder by $3x^4$, we obtain $-2x$ for the second term of the root. Cubing x^2-2x, writing it below, and subtracting, we have the second remainder. Dividing the first term of this remainder again by $3x^4$, we obtain a^2 for the third term of the root. The cube of x^2-2x+a^2 being equal to the given polynomial, the work is finished.

REMARKS.—1. A second method for extracting the square root, similar to the above, might be given, but it is less simple than the common rule.

2. The process of cubing the root may be conducted by Newton's Theorem, as explained in Art. 172.

Find the cube root

2. Of $a^3+24a^2b+192ab^2+512b^3$. Ans. $a+8b$.

3. Of $8a^3-84a^2x+294ax^2-343x^3$. Ans. $2a-7x$.

4. Of $a^6-6a^5+15a^4-20a^3+15a^2-6a+1$.

Ans a^2-2a+1.

5. Of $x^6-9x^5+39x^4-99x^3+156x^2-144x+64$.

Ans. x^2-3x+4.

6. Of $(a+1)^{6n}x^3-6ca^p(a+1)^{4n}x^2+12c^2a^{2p}(a+1)^{2n}x-8c^3a^{3p}$. Ans. $(a+1)^{2n}x-2ca^p$.

7. Find the first three terms of the cube root of $1-x$.

Ans. $1-\frac{x}{3}-\frac{x^2}{9}-$, etc.

IV. EXTRACTION OF THE FOURTH ROOT, SIXTH ROOT, N^{TH} ROOT, ETC.

192. The fourth root of a number is one of four equal factors, into which the number may be resolved; and, in general, the n^{th} root of a number is one of the n equal factors into which the number may be resolved.

When the degree of the root to be extracted is a multiple of two or more numbers, as 4, 6, etc., *the root can be obtained by extracting the roots of more simple degrees.*

To explain this, we remark that $(a^3)^4=a^{3\times4}=a^{12}$, and in general $(a^m)^n=a^{m\times n}=a^{mn}$. Hence,

The n^{th} *power of the* m^{th} *power of a number is equal to the* mn^{th} *power of the number.*

Reciprocally, *the* mn^{th} *root of a number, is equal to the* n^{th} *root of the* m^{th} *root of that number;* that is,

$$\sqrt[mn]{a}=\sqrt[n]{\sqrt[m]{a}}.$$

From this, it follows that $\sqrt[4]{a}=\sqrt{\sqrt{a}}$; and $\sqrt[6]{a}=\sqrt[3]{\sqrt{a}}$, or $\sqrt{\sqrt[3]{a}}$; in like manner $\sqrt[8]{a}=\sqrt{\sqrt{\sqrt{a}}}$, and so on.

1. Find the 4th root of 65536. . . . . . Ans. 16.
2. Find the 4th root of 13107.9601. . . Ans. 10.7.
3. Find the 6th root of 2985984. . . . Ans. 12.
4. Find the 8th root of 214358881. . . Ans. 11.
5. Find the 4th root of $81a^4x^8$. . . . . Ans. $3ax^2$.
6. Find the 4th root of $a^4+4a^3bx+6a^2b^2x^2+4ab^3x^3+b^4x^4$. Ans. $a+bx$.
7. Find the 4th root of $x^8-4x^6+10x^4-16x^2+19-\frac{16}{x^2}+\frac{10}{x^4}-\frac{4}{x^6}+\frac{1}{x^8}$. Ans. $x^2-1+\frac{1}{x^2}$.
8. Find the 6th root of $a^6+\frac{1}{a^6}-6\left(a^4+\frac{1}{a^4}\right)+15\left(a^2+\frac{1}{a^2}\right)-20$. Ans. $a-\frac{1}{a}$.

193. It has been shown already (Arts. 182, 183) that the square root of a monomial, or a polynomial, may be preceded either by the sign $+$ or $-$. We shall now explain the law in regard to the roots generally.

If we take the successive powers of $+a$ and $-a$, we have

$$+a,\quad +a^2,\quad +a^3,\quad +a^4 \ldots\ldots$$
$$-a,\quad +a^2,\quad -a^3,\quad +a^4,\ \ldots\ +a^{2n},\quad -a^{2n+1}.$$

From this we see that every *even* power is positive, and that an *odd* power has the same sign as the root.

Conversely, it is evident,

1st. That *every odd root of a monomial must have the same sign as the monomial itself.*

Thus, $\sqrt[3]{+8a^3}=+2a$, $\sqrt[3]{-8a^3}=-2a$, $\sqrt[5]{-32a^{10}}=-2a^2$.

2d. That *an even root of a positive monomial may be either positive or negative.*

Thus, $\sqrt[4]{81a^4b^8}=\pm 3ab^2$, $\sqrt[6]{64a^{12}}=\pm 2a^2$.

3d. That *every even root of a negative monomial is impossible;* since no quantity raised to a power of an even degree can give a negative result.

Thus, $\sqrt[4]{-a^4}$, $\sqrt[6]{-b}$, $\sqrt[8]{-c}$, are symbols of operations which can not be performed. They are imaginary expressions, like $\sqrt{-a}$, $\sqrt{-b}$, (Art. 182.)

TO EXTRACT THE N^{TH} ROOT OF ANY QUANTITY.

194. In raising any monomial to the n^{th} power, (Art. 172,) we raise the numeral coëfficient to the n^{th} power, and multiply each exponent by n, thus, $(2a^2b^4)^3=8a^6b^{12}$.

Hence, conversely, to find the n^{th} root of a monomial,

Extract the nth *root of the coëfficient, and divide the exponent of each letter by* n.

Rules for the extraction of any root of a numerical quantity, or algebraic polynomial, may be formed on the same principle as is that of the cube root, (Art. 191.) Thus, since

$(a+b)^4=a^4+4a^3b+6a^2b^2+4ab^3+b^4=a^4+(4a^3+6a^2b+4ab^2+b^3)b$.

$(a+b)^5=a^5+(5a^4+10a^3b+10a^2b^2+5ab^3+b^4)b$, etc.

The trial divisor for the fourth root would be of the form $4a^3$, or four times the third power of the first term of the root, and the complete divisor of the form, $4a^3+6a^2b+4ab^2+b^3$.

For the fifth root, the trial and complete divisors would be of the forms, $5a^4$ and $5a^4+10a^3b+10a^2b^2+5ab^3+b^4$, and so for any higher root.

A more simple method, however, would be like that which is called the *Second Method* for extracting the cube root, (Art. 191.) The trial divisors would be of the form $4a^3$, for the 4th root, $5a^4$ for the 5th root, na^{n-1} for the n^{th} root, or, in general, n *times the* $(n-1)^{th}$ *power of the first term of the root.*

Remark.—In the following examples, find the root of the numeral coëfficient by inspection. It is unnecessary to give rules for extracting the 5th, 7th, etc., roots of numbers, as in the present state of science these operations are readily performed by Logarithms.

1. Find the 5th root of $-32a^5x^{10}$. . . . Ans. $-2ax^2$.
2. The 6th root of $729b^6c^{18}$. . . . Ans. $\pm 3bc^3$.
3. The 7th root of $128x^7y^{14}$. . . . . Ans. $2xy^2$.
4. The 8th root of $6561a^8b^{16}$. . . . Ans. $\pm 3ab^2$.
5. The 9th root of $-512x^9z^{18}$. . . . Ans. $-2xz^2$.
6. The 10th root of $1024b^{10}z^{30}$. . . Ans. $\pm 2bz^3$.
7. Extract $\sqrt[n]{a^{4n}b^nc^{2n}}$. . . . . Ans. a^4bc^2.
8. Extract the 5th root of $32x^5-80x^4+80x^3-40x^2+10x-1$. Ans. $2x-1$.

V. RADICAL QUANTITIES.

NOTE.—These quantities are generally called *surds* by English writers; while the French more properly term them *radicals*, from the Latin word *radix*, a root.

195. A **Rational Quantity** is either not affected by the radical sign, or the root indicated can be exactly ascertained; thus, 2, a, $\sqrt{4}$, and $\sqrt[3]{8}$ are rational quantities.

A **Radical Quantity** is one affected by a radical sign, but whose indicated root can not be exactly expressed in numbers; thus, $\sqrt{5}=2.23606797$ nearly.

196. From Art. 193 it is evident that when a monomial is a perfect power of the n^{th} degree, its numeral coëfficient is a perfect power of that degree, and the exponent of each letter is divisible by n.

Thus, $4a^2$ is a perfect square, while $6a^3$ is not; and $8a^6$ is a perfect cube, while $6a^5$, $8a^2$, $7a^6$, etc., are not.

In extracting any root, when the exact division of the exponent can not be performed, it may be indicated by a fraction. Thus, $\sqrt{a^3}$ may be written $a^{\frac{3}{2}}$, and $\sqrt[3]{a^4}$ may be written $a^{\frac{4}{3}}$; and, in general, the n^{th} root of the m^{th} power of a is either $\sqrt[n]{a^m}$, or $a^{\frac{m}{n}}$.

Since a is the same as a^1, (Art 19,) the square root of a may be expressed thus, $a^{\frac{1}{2}}$; the cube root thus, $a^{\frac{1}{3}}$; and the n^{th} root thus, $a^{\frac{1}{n}}$. Hence, the following expressions are equivalent:

$\sqrt{a}$ and $a^{\frac{1}{2}}$,	Also, $\sqrt[3]{a^2}$ and $a^{\frac{2}{3}}$,
$\sqrt[3]{a}$ and $a^{\frac{1}{3}}$,	$\sqrt[n]{a^m}$ and $a^{\frac{m}{n}}$
$\sqrt[n]{a}$ and $a^{\frac{1}{n}}$.	Hence,

The numerator of the fractional exponent denotes the power of the quantity, and the denominator the root to be extracted.

197. Theorem.—*Any quantity affected with a fractional exponent, may be transferred from one term of a fraction to the other, if, at the same time, the sign of its exponent be changed.*

This proposition has already been established (Art. 81) when the exponent is integral. It is also true when the exponent is fractional, as we shall now prove.

Let it be required to extract the cube root of $\frac{1}{a^2}$.

As $\frac{1}{a^2}=a^{-2}$ (Art. 81); therefore, $\sqrt[3]{\frac{1}{a^2}}=\sqrt[3]{a^{-2}}$.

But, $\sqrt[3]{\frac{1}{a^2}}=\frac{1}{a^{\frac{2}{3}}}$ and $\sqrt[3]{a^{-2}}=a^{-\frac{2}{3}}$. (Arts. 190, 194.)

Therefore, $\frac{1}{a^{\frac{2}{3}}}=a^{-\frac{2}{3}}$.

In like manner, generally, $\frac{1}{a^{\frac{m}{n}}}=a^{-\frac{m}{n}}$

198. The **Coefficient** of the radical is the quantity which stands before the radical sign.

Thus, in the expressions $a\sqrt{b}$, and $2\sqrt[3]{c}$, the quantities a and 2 are called coëfficients.

Radicals are said to be of the *same degree* when they have the same index; as, $a^{\frac{2}{3}}$ and $5^{\frac{2}{3}}$, or $\sqrt[3]{a^2}$ and $\sqrt[3]{5^2}$.

Similar radicals have the same index, and the same quantity under the radical sign; as, $a\sqrt{b}$ and $c\sqrt{b}$; $3\sqrt[3]{a^2}$ and $5\sqrt[3]{a^2}$.

Before entering into a discussion of the general subject of radicals, it is important to observe that,

A radical quantity is raised to a power equal to the index of its root, by simply rejecting the radical sign with its index.

Thus, the square of $\sqrt{a}$ is a, the cube of $\sqrt[3]{a}$ is a, the square of $\sqrt{3}$ is 3, the n^{th} power of $\sqrt[n]{a}$ is a, etc. In other words, $\sqrt{a}\times\sqrt{a}=a$, $\sqrt[3]{a}\times\sqrt[3]{a}\times\sqrt[3]{a}=a$, etc. This is evident from the definition of a root, (Art. 173.)

REDUCTION OF RADICALS.

Case I.—To reduce Radicals to their Simplest Form.

199. Reduction of radicals consists in changing the form of the quantities without altering their value. It is founded on the following principle:

The square root of the product of two or more factors is equal to the product of the square roots of those factors.

That is, $\sqrt{ab}=\sqrt{a}\times\sqrt{b}$; which is thus proved;

Squaring both members of this equation, we have, (Art. 198,)

$$ab=a\times b, \text{ or } ab=ab.$$

Now, since the equation is true after both sides are squared, it was true before, (Art. 148, Ax. 6,) or $\sqrt{ab}=\sqrt{a}\times\sqrt{b}$.

By this principle, $\sqrt{36}=\sqrt{4\times9}=2\times3$; $\sqrt{144}=\sqrt{9\times16}=3\times4$; $\sqrt{8}=\sqrt{4\times2}=\sqrt{4}\times\sqrt{2}=2\sqrt{2}$. Hence, we have the following

Rule for the Reduction of a Radical of the Second Degree to its Simplest Form.—1st. *Separate the quantity to be reduced into two parts, one of which shall contain all the factors that are perfect squares, and the other the remaining factors.*

2d. *Extract the square root of the perfect square, and prefix it as a coëfficient to the other part placed under the radical sign.*

To determine whether any numeral contains a factor that is a perfect square, divide it by either of the squares 4, 9, 16, etc.

Reduce to their simplest forms the radicals in each of the following examples:

1. $\sqrt{12}$, $\sqrt{18}$, $\sqrt{45}$, $\sqrt{32}$, $\sqrt{50a^3}$, $\sqrt{72a^2b^3}$.

   Ans. $2\sqrt{3}$, $3\sqrt{2}$, $3\sqrt{5}$, $4\sqrt{2}$, $5a\sqrt{2a}$, $6ab\sqrt{2b}$.

2. $\sqrt{245}$, $\sqrt{448}$, $\sqrt{810}$, $\sqrt{507b^3c^2}$, $\sqrt{1805a^4b^2}$.

   Ans. $7\sqrt{5}$, $8\sqrt{7}$, $9\sqrt{10}$, $13bc\sqrt{3b}$, $19a^2b\sqrt{5}$.

In a similar manner, polynomials may sometimes be simplified. Thus, $\sqrt{(3a^3-6a^2c+3ac^2)}=\sqrt{3a(a^2-2ac+c^2)}=(a-c)\sqrt{3a}$.

3. $\sqrt{(a^3-a^2b)}$, $\sqrt{ax^2-6ax+9a}$, $\sqrt{(x^2-y^2)(x+y)}$.

   Ans. $a\sqrt{(a-b)}$, $(x-3)\sqrt{a}$, $(x+y)\sqrt{(x-y)}$.

To reduce a fractional radical to its simplest form,

1st. *Render the denominator of the fraction a perfect square by multiplying or dividing both terms by the same quantity.*

2d. *Separate into two factors, one of which is a perfect square.*

3d. *Extract the square root of this factor, and write it as a coëfficient to the other factor placed under the radical sign.*

4. Reduce $\sqrt{\frac{4}{5}}$, and $\sqrt{\frac{a}{b}}$, to their simplest forms.

$$\sqrt{\tfrac{4}{5}}=\sqrt{\tfrac{4}{5}\times\tfrac{5}{5}}=\sqrt{\tfrac{20}{25}}=\sqrt{\tfrac{4}{25}\times5}=\sqrt{\tfrac{4}{25}}\times\sqrt{5}=\tfrac{2}{5}\sqrt{5}.$$

$$\sqrt{\frac{a}{b}}=\sqrt{\frac{a}{b}\times\frac{b}{b}}=\sqrt{\frac{ab}{b^2}}=\sqrt{\frac{1}{b^2}\times ab}=\sqrt{\frac{1}{b^2}}\times\sqrt{ab}=\frac{1}{b}\sqrt{ab}.$$

5. $\sqrt{\frac{1}{2}}$, $\sqrt{\frac{3}{8}}$, $\sqrt{\frac{18}{25}}$, $6\sqrt{\frac{1}{12}}$, $30\sqrt{\frac{3}{10}}$, $18\sqrt{\frac{5}{72}}$.

   Ans. $\frac{1}{2}\sqrt{2}$, $\frac{1}{4}\sqrt{6}$, $\frac{3}{5}\sqrt{2}$, $\sqrt{3}$, $3\sqrt{30}$, $\frac{3}{2}\sqrt{10}$.

6. $\sqrt{\frac{c^2}{b}}$, $\sqrt{\frac{3a}{5b}}$, $\sqrt{\frac{a^3x^2}{4c^2y}}$, $\left(\frac{3xy^3}{98z^4}\right)^{\frac{1}{2}}$.

Ans. $\frac{c}{b}\sqrt{b}$, $\frac{1}{5b}\sqrt{15ab}$, $\frac{ax}{2cy}\sqrt{ay}$, $\frac{y}{14z^2}(6xy)^{\frac{1}{2}}$.

200. To reduce radicals of any degree to the most simple form.

The principle of Art. 199 is, evidently, applicable to radicals of any degree. Thus,

1. Reduce $\sqrt[3]{54}$ to its most simple form.

$$\sqrt[3]{54}=\sqrt[3]{27\times2}=\sqrt[3]{27}\times\sqrt[3]{2}=3\sqrt[3]{2}.$$

Similarly, $\sqrt[3]{\frac{2}{3}}=\sqrt[3]{\frac{2}{3}\times\frac{3}{3}\times\frac{3}{3}}=\sqrt[3]{\frac{18}{27}}=\sqrt[3]{\frac{1}{27}\times18}=\frac{1}{3}\sqrt[3]{18}$.

Reduce each of the following to its simplest form:

2. $\sqrt[3]{40}$, $\sqrt[3]{81c^4}$, $\sqrt[3]{128a^6c^5}$, $\sqrt[3]{162m^4n^5}$, $\sqrt[4]{144}$.

Ans. $2\sqrt[3]{5}$, $3c\sqrt[3]{3c}$, $4a^2c\sqrt[3]{2c^2}$, $3mn\sqrt[3]{6mn^2}$, $2\sqrt[4]{9}$.

3. $\sqrt[3]{\frac{1}{2}}$, $\sqrt[3]{\frac{3}{4}}$, $\sqrt[3]{\frac{1}{6}}$, $\sqrt[3]{\frac{5}{9}}$, $\sqrt[4]{\frac{2}{3}}$, $\sqrt[5]{\frac{3}{4}}$, $\sqrt[6]{\frac{1}{2}}$.

Ans. $\frac{1}{2}\sqrt[3]{4}$, $\frac{1}{2}\sqrt[3]{6}$, $\frac{1}{6}\sqrt[3]{36}$, $\frac{1}{3}\sqrt[3]{15}$, $\frac{1}{3}\sqrt[4]{54}$, $\frac{1}{4}\sqrt[5]{768}$, $\frac{1}{2}\sqrt[6]{32}$.

4. $\sqrt[4]{162}$, $\sqrt[4]{3888}$, $\sqrt[4]{32a^5b^7}$, $\sqrt[5]{729a^6}$.

Ans. $3\sqrt[4]{2}$, $6\sqrt[4]{3}$, $2ab\sqrt[4]{2ab^3}$, $3a\sqrt[5]{3a}$.

201. The mn^{th} root of any quantity may be simplified when it is a complete power of the m^{th} or n^{th} degree, as shown, (Art. 192.)

Thus, $\sqrt[4]{9a^2}=\sqrt{\sqrt{9a^2}}=\sqrt{3a}$.

Also, $\sqrt[6]{a^2-2ab+b^2}=\sqrt[3]{\sqrt{a^2-2ab+b^2}}=\sqrt[3]{a-b}$.

Reduce each of the following to its simplest form:

1. $\sqrt[4]{36a^2c^2}$, $\sqrt[4]{81m^2n^4}$, $\sqrt[6]{4a^2}$, $\sqrt[6]{16a^2c^4}$, $\sqrt[6]{125b^3}$.

Ans. $\sqrt{6ac}$, $3n\sqrt{m}$, $\sqrt[3]{2a}$, $\sqrt[3]{4ac^2}$, $\sqrt{5b}$.

Case II.—To reduce a Rational Quantity to the form of a Radical.

202. If we square a, and then extract the square root of the square, the result is evidently a.

That is, $a=\sqrt{a^2}=a^{\frac{2}{2}}$. In like manner, $a=\sqrt[3]{a^3}=a^{\frac{3}{3}}$, and generally, $a=\sqrt[m]{a^m}=a^{\frac{m}{m}}$ Hence,

Rule for reducing a Rational Quantity to the form of a Radical.—*Raise the quantity to a power corresponding to the given root, and write it under the radical sign.*

1. Reduce 6 to the form of the square root. Ans. $\sqrt{36}$.
2. -2 to the form of the cube root. Ans. $\sqrt[3]{-8}$.
3. $3ax$ to the form of the square root. Ans. $\sqrt{9a^2x^2}$.
4. $m-n$ to the form of the square root. Ans. $\sqrt{m^2-2mn+n^2}$.

Similarly, a coefficient may be passed under the radical sign.

Thus, $2\sqrt{3}=\sqrt{4}\times\sqrt{3}=\sqrt{12}$.

Generally, $a\sqrt[m]{b}=\sqrt[m]{a^m}\times\sqrt[m]{b}=\sqrt[m]{a^mb}$.

5. Express $5\sqrt{7}$, and $a^2\sqrt{b}$, entirely under the radical sign. Ans. $\sqrt{175}$, and $\sqrt{a^4b}$.

6. Pass the coefficient of the quantity $2\sqrt[3]{5}$, under the radical sign. Ans. $\sqrt[3]{40}$.

Case III.—To reduce Radicals having Different Indices to Equivalent Radicals having a Common Index.

203. This is done by multiplying both terms of the fractional exponent by the same number, which, evidently, does not change its value. (Art. 118.)

Let it be required to reduce $\sqrt[3]{2a}$, and $\sqrt[4]{3b}$, or $(2a)^{\frac{1}{3}}$ and $(3b)^{\frac{1}{4}}$ to quantities of equal value, having the same index.

$$\sqrt[3]{2a}=(2a)^{\frac{1}{3}}=(2a)^{\frac{4}{12}}=\sqrt[12]{(2a)^4}=\sqrt[12]{16a^4}.$$

$$\sqrt[4]{3b}=(3b)^{\frac{1}{4}}=(3b)^{\frac{3}{12}}=\sqrt[12]{(3b)^3}=\sqrt[12]{27b^3}.\quad \text{Hence,}$$

Rule.—*Reduce the fractional exponents to a common denominator; then the numerator of each fraction will represent the power to which the corresponding quantity is to be raised, and the common denominator the index of the root to be extracted.*

1. Reduce $\sqrt{3}$ and $\sqrt[3]{2}$, or $3^{\frac{1}{2}}$ and $2^{\frac{1}{3}}$ to a common index.

Ans. $\sqrt[6]{27}$ and $\sqrt[6]{4}$, or $27^{\frac{1}{6}}$ and $4^{\frac{1}{6}}$.

2. $\sqrt[3]{5}$ and $\sqrt{4}$. . . . . Ans. $\sqrt[6]{25}$ and $\sqrt[6]{64}$.

3. a^2 and $b^{\frac{1}{2}}$. . . . . Ans. $\sqrt{a^4}$ and $\sqrt{b}$.

4. $\sqrt[4]{a}$, $\sqrt[6]{5b}$, and $\sqrt[8]{6c}$.

Ans. $\sqrt[24]{a^6}$, $\sqrt[24]{625b^4}$, and $\sqrt[24]{216c^3}$.

5. $\sqrt{a^3}$, $\sqrt[3]{a^2}$, and $\sqrt[4]{a^3}$. Ans. $\sqrt[12]{a^{18}}$, $\sqrt[12]{a^8}$, and $\sqrt[12]{a^9}$.

6. Reduce $3^{\frac{2}{3}}$, $2^{\frac{3}{4}}$, and $5^{\frac{1}{2}}$ to a common index.

Ans. $3^{\frac{8}{12}}$, $2^{\frac{9}{12}}$, $5^{\frac{6}{12}}$, or $\sqrt[12]{6561}$, $\sqrt[12]{512}$, $\sqrt[12]{15625}$.

ADDITION AND SUBTRACTION OF RADICALS.

204. Required to find the sum of $3\sqrt[3]{a}$ and $5\sqrt[3]{a}$.

It is evident that 3 times and 5 times any quantity, must make 8 times that quantity; therefore, $3\sqrt[3]{a}+5\sqrt[3]{a}=8\sqrt[3]{a}$.

But, if it were required to find the sum of $3\sqrt{a}$ and $5\sqrt[3]{a}$, since $\sqrt{a}$ and $\sqrt[3]{a}$ are *different* quantities, we can only *indicate* their addition; thus, $3\sqrt{a}+5\sqrt[3]{a}$.

Similarly, $3\sqrt{2}+7\sqrt{2}-4\sqrt{2}=6\sqrt{2}$.

But $3\sqrt{5}$ and $4\sqrt{3}=3\sqrt{5}+4\sqrt{3}$.

So also $3\sqrt{5}$ and $4\sqrt[3]{5}=3\sqrt{5}+4\sqrt[3]{5}$.

Radicals that are not similar, may often be made so; thus, $\sqrt{12}$ and $\sqrt{27}$ are equal to $2\sqrt{3}$ and $3\sqrt{3}$, and their sum is $5\sqrt{3}$.

The same principles apply to the subtraction of radicals.

From the above we derive the following

Rule for the Addition of Radicals.—1st. *Reduce the radicals to their simplest forms, and, if necessary, to a common index.*

2d. *If the radicals are similar, find the sum of their coëfficients, and prefix it to the common radical; but if they are not similar, connect them by their proper signs.*

Rule for the Subtraction of Radicals.—*Change the sign of the subtrahend, and proceed as in addition of radicals.*

1. Find the sum of $\sqrt{448}$ and $\sqrt{112}$.

$$\sqrt{448}=\sqrt{64\times7}=8\sqrt{7}$$
$$\sqrt{112}=\sqrt{16\times7}=4\sqrt{7}$$

By addition, $12\sqrt{7}$, Ans.

2. Find the sum of $\sqrt[3]{24}$ and $\sqrt[3]{81}$. . Ans. $5\sqrt[3]{3}$.
3. Of $\sqrt[3]{48}$ and $\sqrt[3]{162}$. . . . Ans. $5\sqrt[3]{6}$.
4. Of $\sqrt{18a^5b^3}$ and $\sqrt{50a^3b^3}$.

Ans. $(3a^2b+5ab)\sqrt{2ab}$.

5. Subtract $\sqrt{180}$ from $\sqrt{405}$. . . . Ans. $3\sqrt{5}$.
6. Subtract $\sqrt[3]{40}$ from $\sqrt[3]{135}$. . . . . Ans. $\sqrt[3]{5}$.

Perform the operations indicated in each of the following:

7. $\sqrt{243}+\sqrt{27}+\sqrt{48}$. . . . . Ans. $16\sqrt{3}$.
8. $\sqrt{24}+\sqrt{54}-\sqrt{96}$. . . . . Ans. $\sqrt{6}$.
9. $\sqrt{128}-2\sqrt{50}+\sqrt{72}-\sqrt{18}$. . . . . Ans. $\sqrt{2}$.

10. $\sqrt{48ab^2}+b\sqrt{75a}+\sqrt{3a(a-9b)^2}$. . . . . . Ans. $a\sqrt{3a}$.
11. $2\sqrt{\frac{5}{3}}+\frac{1}{6}\sqrt{60}+\sqrt{15}+\sqrt{\frac{3}{5}}$. . . . . Ans. $\frac{11}{5}\sqrt{15}$.
12. $\sqrt[3]{128}-\sqrt[3]{686}-\sqrt[3]{16}+4\sqrt[3]{250}$. . . . . Ans. $15\sqrt[3]{2}$.
13. $2\sqrt[3]{\frac{1}{4}}+8\sqrt[3]{\frac{1}{32}}$. . . . . Ans. $3\sqrt[3]{2}$.
14. $6\sqrt[6]{4a^2}+2\sqrt[3]{2a}+\sqrt[9]{8a^3}$. . . . . Ans. $9\sqrt[3]{2a}$.
15. $2\sqrt{3}-\frac{1}{2}\sqrt{12}+4\sqrt{27}-2\sqrt{\frac{3}{16}}$. . . . . . . Ans. $\frac{25}{2}\sqrt{3}$.
16. $\sqrt[4]{16}+\sqrt[3]{81}-\sqrt[3]{-512}+\sqrt[3]{192}-7\sqrt[6]{9}$. . . . . Ans. 10.
17. $\sqrt{\frac{ab^3}{c^2}}+\frac{1}{2c}\sqrt{(a^3b-4a^2b^2+4ab^3)}$. . . . . Ans. $\frac{a}{2c}\sqrt{ab}$.

MULTIPLICATION AND DIVISION OF RADICALS.

205. The rule for the multiplication of radicals is founded on the principle (Art. 200) that

The product of the n*th* *root of two or more quantities is equal to the* n*th* *root of their product.*

That is, $\sqrt[n]{a}\times\sqrt[n]{b}=\sqrt[n]{ab}$. (See Art. 198.)
Hence, (Art. 53,) $a\sqrt[n]{b}\times c\sqrt[n]{d}=a\times c\times\sqrt[n]{b}\times\sqrt[n]{d}=ac\sqrt[n]{bd}$.

The rule for division is founded on the principle that

The quotient of the n*th* *roots of two quantities is equal to the* n*th* *root of their quotient.*

That is, $\frac{\sqrt[n]{a}}{\sqrt[n]{b}}=\sqrt[n]{\frac{a}{b}}$; which is thus proved:

Raising both sides to the *nth* power, we have $\frac{a}{b}=\frac{a}{b}$, which shows that the previous equation is true. Hence, we have the following

Rules for the Multiplication and Division of Radicals.—*If the radicals have different indices, reduce them to the same index. Then,*

I. **To Multiply.**—*Multiply the coëfficients together for the coefficient of the product, and also the parts under the radical for the radical part of the product.*

II. **To Divide.**—*Divide the coëfficient of the dividend by the coëfficient of the divisor for the coëfficient of the quotient, and the radical part of the dividend by the radical part of the divisor for the radical part of the quotient.*

1. Multiply $2\sqrt{ab}$ by $3a\sqrt{abc}$.

$$\begin{array}{l} 2\sqrt{ab} \\ 3a\sqrt{abc} \\ \hline 6a\sqrt{a^2b^2c}=6a\sqrt{a^2b^2\times c}=6a\times ab\sqrt{c}=6a^2b\sqrt{c}. \end{array}$$

2. Divide $4a\sqrt{ab}$ by $2\sqrt{ac}$.

$$\frac{4a\sqrt{ab}}{2\sqrt{ac}}=\frac{4a}{2}\sqrt{\frac{ab}{ac}}=2a\sqrt{\frac{b}{c}}=2a\sqrt{\frac{bc}{c^2}}=\frac{2a}{c}\sqrt{bc}.$$

3. Multiply $2\sqrt[3]{3}$ by $3\sqrt{2}$.

$$2\sqrt[3]{3}=2(3)^{\frac{1}{3}}=2(3)^{\frac{2}{6}}=2\sqrt[6]{3^2}=2\sqrt[6]{9}.$$
$$3\sqrt{2}=3(2)^{\frac{1}{2}}=3(2)^{\frac{3}{6}}=3\sqrt[6]{2^3}=3\sqrt[6]{8}.$$

Multiplying, . . $6\sqrt[6]{72}$, **Ans.**

4. Divide $6\sqrt{2}$ by $3\sqrt[3]{2}$.

$$6\sqrt{2}=6\sqrt[6]{2^3}=6\sqrt[6]{8}. \qquad (1.)$$
$$3\sqrt[3]{2}=3\sqrt[6]{2^2}=3\sqrt[6]{4}. \qquad (2.)$$

Dividing (1) by (2), we have $2\sqrt[6]{2}$.

5. Multiply $3\sqrt{12}$ by $5\sqrt{18}$. . . . Ans. $90\sqrt{6}$.
6. Multiply $4\sqrt[3]{12}$ by $3\sqrt[3]{4}$. . . . . Ans. $24\sqrt[3]{6}$.
7. Multiply together $5\sqrt{3}$, $7\sqrt{\frac{8}{3}}$, and $\sqrt{2}$. Ans 140
8. Multiply $3\sqrt[3]{b}$ by $4\sqrt[4]{a}$. . . Ans. $12\sqrt[12]{a^3b^4}$.
9. Multiply together $\sqrt{2}$, $\sqrt[3]{3}$, and $\sqrt[4]{5}$. A. $\sqrt[12]{648000}$.
10. Multiply together $\sqrt[2n]{x}$, $\sqrt[n]{x^2}$, and $\sqrt[3n]{x^3}$. Ans. $\sqrt[2n]{x^7}$.
11. Divide $\sqrt{40}$ by $\sqrt{2}$. . . . . . . Ans. $2\sqrt{5}$.
12. Divide $6\sqrt{54}$ by $3\sqrt{2}$. . . . . . . Ans. $6\sqrt{3}$.

13. Divide $70\sqrt[3]{9}$ by $7\sqrt[3]{18}$. . . . . . . Ans. $5\sqrt[3]{4}$.
14. Divide $\sqrt[6]{72}$ by $\sqrt{2}$. . . . . Ans. $\sqrt[3]{3}$.
15. Divide $4\sqrt[3]{9}$ by $2\sqrt{3}$. . . . . Ans. $2\sqrt[6]{3}$.
16. Divide $\sqrt[6]{72}$ by $\sqrt[3]{3}$. . . . . Ans. $\sqrt{2}$.
17. Divide $\sqrt[4]{\frac{b}{a}}$ by $\sqrt[4]{\frac{a}{b}}$. . . . . Ans. $\sqrt{\frac{b}{a}}$.

Polynomials containing radicals may also be multiplied; thus,

18. Multiply $3+\sqrt{5}$ by $2-\sqrt{5}$.

$$\begin{array}{l} 3+\ \sqrt{5} \\ 2-\ \sqrt{5} \\ \hline 6+2\sqrt{5} \\ \quad -3\sqrt{5}-5 \\ \hline 6-\ \sqrt{5}-5=1-\sqrt{5},\text{ Ans.} \end{array}$$

19. Multiply $\sqrt{2}+1$ by $\sqrt{2}-1$. . . . . . Ans. 1.
20. $11\sqrt{2}-4\sqrt{15}$ by $\sqrt{6}+\sqrt{5}$. Ans. $2\sqrt{3}-\sqrt{10}$.
21. Raise $\sqrt{2}+\sqrt{3}$ to the 4th power. Ans. $49+20\sqrt{6}$.
22. Multiply $\sqrt[3]{12+\sqrt{19}}$ by $\sqrt[3]{12-\sqrt{19}}$. Ans. 5.
23. Multiply $x^2-x\sqrt{2}+1$ by $x^2+x\sqrt{2}+1$. Ans. x^4+1.
24. $(x^2+1)(x^2-x\sqrt{3}+1)(x^2+x\sqrt{3}+1)$. Ans. x^6+1.

206. To reduce a fraction whose denominator contains radicals, to an equivalent fraction having a rational denominator.

When the denominator is a monomial, as $\frac{a}{\sqrt{b}}$, it will become rational if we multiply both terms by $\sqrt{b}$.

$$\text{Thus, } \frac{a}{\sqrt{b}}=\frac{a\times\sqrt{b}}{\sqrt{b}\times\sqrt{b}}=\frac{a\sqrt{b}}{b}.$$

Again, if the denominator is $\sqrt[3]{a}$, if we multiply both terms by $\sqrt[3]{a^2}$, the denominator will become $\sqrt[3]{a}\times\sqrt[3]{a^2}=\sqrt[3]{a^3}=a$.

In like manner, if the denominator is $\sqrt[m]{a^n}$, it will become rational by multiplying it by $\sqrt[m]{a^{m-n}}$. Therefore,

When the denominator of the fraction is a monomial, multiply both terms by such a factor as will render the exponent of the quantity under the radical equal to the index of the radical.

Since the sum of two quantities, multiplied by their difference, is equal to the difference of their squares (Art. 80); if the fraction is of the form $\frac{a}{b+\sqrt{c}}$, and we multiply both terms by $b-\sqrt{c}$, the denominator will be rational.

$$\text{Thus, } \frac{a}{b+\sqrt{c}}=\frac{a(b-\sqrt{c})}{(b+\sqrt{c})(b-\sqrt{c})}=\frac{ab-a\sqrt{c}}{b^2-c}.$$

If the denominator is $b-\sqrt{c}$, the multiplier will be $b+\sqrt{c}$. If the denominator is $\sqrt{b}+\sqrt{c}$, the multiplier will be $\sqrt{b}-\sqrt{c}$; and if it is $\sqrt{b}-\sqrt{c}$, the multiplier will be $\sqrt{b}+\sqrt{c}$.

If the denominator is of the form $\sqrt{a}+\sqrt{b}+\sqrt{c}$, it may be rendered rational by two successive multiplications. The first will result in a quantity of the form $m-\sqrt{n}$, which may be made rational as before.

Reduce the following fractions to equivalent ones having rational denominators:

1. $\frac{1}{\sqrt{3}}$. Ans. $\frac{\sqrt{3}}{3}=\frac{1}{3}\sqrt{3}$.

2. $\frac{\sqrt{3}}{\sqrt{6}}$. Ans. $\frac{\sqrt{18}}{6}=\frac{1}{2}\sqrt{2}$.

3. $\frac{2}{\sqrt[3]{3}}$. Ans. $\frac{2}{3}\sqrt[3]{9}$.

4. $\frac{6}{\sqrt[5]{4^3}}$. Ans. $\frac{3}{2}\sqrt[5]{16}$.

5. $\frac{8-5\sqrt{2}}{3-2\sqrt{2}}$. Ans. $4+\sqrt{2}$.

6. $\frac{\sqrt{3}+\sqrt{2}}{\sqrt{3}-\sqrt{2}}$. Ans. $5+2\sqrt{6}$.

7. $\frac{3\sqrt{5}-2\sqrt{2}}{2\sqrt{5}-\sqrt{18}}$. . . . . Ans. $9+\frac{5}{2}\sqrt{10}$.

8. $\frac{3+4\sqrt{3}}{\sqrt{6}+\sqrt{2}-\sqrt{5}}$. . . . . Ans. $\sqrt{6}+\sqrt{2}+\sqrt{5}$.

9. $\dfrac{1}{x+\sqrt{x^2-1}}+\dfrac{1}{x-\sqrt{x^2-1}}$. . . . . Ans. $2x$.

10. $\dfrac{\sqrt{x^2+1}+\sqrt{x^2-1}}{\sqrt{x^2+1}-\sqrt{x^2-1}}+\dfrac{\sqrt{x^2+1}-\sqrt{x^2-1}}{\sqrt{x^2+1}+\sqrt{x^2-1}}$. . . . . Ans. $2x^2$.

REMARK.—By the preceding transformations, the process of finding the numerical value of a fractional radical is very much abridged. Thus, to find the value of $\frac{2}{\sqrt{5}}$, we may divide 2 by the square root of 5, which is 2.2360679+. But $\frac{2}{\sqrt{5}}=\frac{2\sqrt{5}}{5}$, the true value of which is found by multiplying 2.2360679 by 2, and dividing the result by 5.

Reduce each of the following fractions to its simplest form, and find the numerical value of the result:

11. $\frac{2}{\sqrt{5}}$, and $\frac{1}{\sqrt{2}}$. . . . . Ans. .894427+, and .707106+.

12. $\dfrac{\sqrt{20}+\sqrt{12}}{\sqrt{5}-\sqrt{3}}$. . . . . Ans. 15.745966+.

POWERS OF RADICALS.

207. Let it be required to raise $\sqrt[4]{3a}$ to the 3d power.

Taking $\sqrt[4]{3a}$ as a factor three times, we have

$$\sqrt[4]{3a}\times\sqrt[4]{3a}\times\sqrt[4]{3a}=\sqrt[4]{27a^3}.$$

So, $\sqrt[m]{a}\times\sqrt[m]{a}\times\sqrt[m]{a}$ . . to n factors, $=\sqrt[m]{a^n}$. Hence,

Rule for raising a Radical Quantity to any Power.—*Raise the quantity under the radical to the given power, and affect the result with the primitive radical sign.*

If the quantity have a coëfficient, it must also be raised to the given power. Thus, the 4th power of $2\sqrt[3]{3a^2}$ is $16\sqrt[3]{81a^8}$. This, by reduction, becomes $16\sqrt[3]{27a^6\times3a^2}=48a^2\sqrt[3]{3a^2}$.

If the index of the radical is a multiple of the exponent of the power, the operation may be simplified. Thus,

$$(\sqrt[4]{2a})^2=(\sqrt{\sqrt{2a}})^2=\sqrt{2a}, \text{ (Art. 192.)}$$

In general, $(\sqrt[mn]{a})^n=\left(\sqrt[n]{\sqrt[m]{a}}\right)^n=\sqrt[m]{a}$. Hence,

If the index of the radical is divisible by the exponent of the power, we may perform this division, and leave the quantity under the radical sign unchanged.

Thus, to raise $\sqrt[8]{3a}$ to the 4th power, we have $\sqrt[8]{81a^4}=\sqrt{\sqrt[4]{81a^4}}=\sqrt{3a}$, or, dividing 8 by 4, we obtain at once $\sqrt{3a}$.

1. Raise $\sqrt[3]{2a}$ to the 4th power. . . . Ans. $2a\sqrt[3]{2a}$.
2. $3\sqrt[3]{2ab^2}$ to the 4th power. . Ans. $162ab^2\sqrt[3]{2ab^2}$.
3. $\sqrt[4]{ac^2}$ to the 2d power. . . . Ans. $c\sqrt{a}$.
4. $\sqrt{ac^2}$ to the 4th power. . . . . Ans. a^2c^4.
5. $\sqrt[6]{3c^2}$ to the 3d power. . . . . . Ans. $c\sqrt{3}$.
6. $\sqrt{x-y}$ to the 3d power.. . Ans. $(x-y)\sqrt{x-y}$.

ROOTS OF RADICALS.

208. Since $\sqrt[m]{\sqrt[n]{a}}=\sqrt[mn]{a}$ (Art. 192), therefore, to extract the roots of radicals, we have the following

Rule.—*Multiply the index of the radical by the index of the root to be extracted, and leave the quantity under the radical sign unchanged.*

Thus, the square root of $\sqrt[3]{2a}$ is $\sqrt{\sqrt[3]{2a}}=\sqrt[6]{2a}$.

If the radical has a coëfficient, its root must also be extracted.

If the quantity under the radical is a perfect power of the same degree as the root to be extracted, the process may be simplified.

Thus, $\sqrt[3]{\sqrt[4]{8a^3}}$ is equal (Art. 192) to $\sqrt[4]{\sqrt[3]{8a^3}}=\sqrt[4]{2a}$.

1. Extract the cube root of $\sqrt{a^2b}$. . . . Ans. $\sqrt[6]{a^2b}$.
2. The 4th root of $16a^8\sqrt[3]{2c}$. . . Ans. $2a^2\sqrt[12]{2c}$.
3. The square root of $\sqrt[3]{49a^2}$. . . . Ans. $\sqrt[3]{7a}$.
4. The cube root of $64\sqrt[4]{8a^6}$. . . . Ans. $4\sqrt[4]{2a^2}$.
5. The cube root of $(m+n)\sqrt{m+n}$. Ans. $\sqrt{m+n}$.

IMAGINARY, OR IMPOSSIBLE QUANTITIES.

209. An imaginary quantity (Arts. 182, 193) is an even root of a negative quantity.

Thus, $\sqrt{-a}$, and $\sqrt[4]{-b^4}$, are imaginary quantities.

The rules for the multiplication and division of radicals (Art. 205) require some modification when imaginary quantities are to be multiplied or divided.

Thus, by the rule (Art. 205), $\sqrt{-a}\times\sqrt{-a}=\sqrt{-a\times-a}=\sqrt{a^2}=\pm a$. But, since the square root of any quantity multiplied by the square root itself, must give the original quantity, (Art. 198,) therefore, $\sqrt{-a}\times\sqrt{-a}=-a$.

210. *Every imaginary quantity may be resolved into two factors, one a real quantity, and the other the imaginary expression, $\sqrt{-1}$, or an expression containing it.*

This is evident, if we consider that every *negative* quantity may be regarded as the product of two factors, one of which is -1. Thus, $-a=a\times-1$, $-b^2=b^2\times-1$, and so on.

Hence, $\sqrt{-a^2}=\sqrt{a^2\times-1}=\sqrt{a^2}\times\sqrt{-1}=\pm a\sqrt{-1}$.

Since the square root of any quantity, multiplied by the square root itself, must give the original quantity;

Therefore, $(\sqrt{-1})^2=\sqrt{-1}\times\sqrt{-1}=-1$.

Also, $(\sqrt{-1})^3=(\sqrt{-1})^2\times\sqrt{-1}=-1\sqrt{-1}=-\sqrt{-1}$.

$(\sqrt{-1})^4=(\sqrt{-1})^2\,(\sqrt{-1})^2=(-1)(-1)=+1$.

Attention to this principle will render all the algebraic operations, with imaginary quantities, easily performed.

Thus, $\sqrt{-a}\times\sqrt{-b}=\sqrt{a}\times\sqrt{-1}\times\sqrt{b}\times\sqrt{-1}=\sqrt{ab}\times(\sqrt{-1})^2=-\sqrt{ab}$.

If it be required to find the product of $a+b\sqrt{-1}$ by $a-b\sqrt{-1}$, the operation is performed as in the margin.

OPERATION.

$$\begin{array}{r} a+b\sqrt{-1} \\ a-b\sqrt{-1} \\ \hline a^2+ab\sqrt{-1} \\ -ab\sqrt{-1}+b^2 \\ \hline a^2+b^2. \end{array}$$

Since $a^2+b^2=(a+b\sqrt{-1})(a-b\sqrt{-1})$, any binomial whose terms are positive may be resolved into two factors, one of which is the sum and the other the difference of a real and an imaginary quantity.

Thus, $m+n=(\sqrt{m}+\sqrt{n}\sqrt{-1})(\sqrt{m}-\sqrt{n}\sqrt{-1})$.

1. Multiply $\sqrt{-a^2}$ by $\sqrt{-b^2}$. . . . Ans. $-ab$.

2. Find the 3d and 4th powers of $a\sqrt{-1}$.

Ans. $-a^3\sqrt{-1}$, and a^4.

3. Multiply $2\sqrt{-3}$ by $3\sqrt{-2}$. . Ans. $-6\sqrt{6}$.

4. Divide $6\sqrt{-3}$ by $2\sqrt{-4}$. . . . Ans. $\frac{3}{2}\sqrt{3}$.

5. Simplify the fraction $\dfrac{1+\sqrt{-1}}{1-\sqrt{-1}}$. . Ans. $\sqrt{-1}$.

6. Find the continued product of $x+a$, $x+a\sqrt{-1}$, $x-a$, and $x-a\sqrt{-1}$. Ans. x^4-a^4.

7. Of what number are $24+7\sqrt{-1}$, and $24-7\sqrt{-1}$, the imaginary factors? Ans. 625.

VI. THEORY OF FRACTIONAL EXPONENTS.

211. The rules for integral exponents in multiplication, division, involution, and evolution, (Arts. 56, 70, 172, and 194.) are equally applicable when the exponents are *fractional.*

Fractional exponents have their origin (Art. 196) in the

extraction of roots, when the exponent of the power is not divisible by the index of the root.

Thus, the cube root of a^2 is $a^{\frac{2}{3}}$. So the n^{th} root of a^m is $a^{\frac{m}{n}}$.

The forms $a^{\frac{2}{3}}$, $a^{\frac{4}{3}}$, and $a^{-\frac{m}{n}}$, may be read a to the power of $\frac{2}{3}$, a to the power of $\frac{4}{3}$, and a to the power of minus $\frac{m}{n}$; or, a exponent $\frac{2}{3}$, a exponent $\frac{4}{3}$, a exponent $-\frac{m}{n}$.

MULTIPLICATION AND DIVISION OF QUANTITIES WITH FRACTIONAL EXPONENTS.

212. It has been shown (Art. 56) that *the exponent of any letter in the product is equal to the sum of its exponents in the two factors.* It will now be shown that the same rule applies when the exponents are fractional.

1. Let it be required to multiply $a^{\frac{2}{3}}$ by $a^{\frac{4}{5}}$.

$$a^{\frac{2}{3}}=\sqrt[3]{a^2}=\sqrt[15]{a^{10}},\quad a^{\frac{4}{5}}=\sqrt[5]{a^4}=\sqrt[15]{a^{12}},\ \text{(Art. 205.)}$$

$$a^{\frac{2}{3}}\times a^{\frac{4}{5}}=\sqrt[15]{a^{10}}\times\sqrt[15]{a^{12}}=\sqrt[15]{a^{10+12}}=\sqrt[15]{a^{22}}=a^{\frac{22}{15}}.$$

But this result is the same as that obtained by adding the exponents together.

$$\text{Thus, } a^{\frac{2}{3}}\times a^{\frac{4}{5}}=a^{\frac{2}{3}+\frac{4}{5}}=a^{\frac{10}{15}+\frac{12}{15}}=a^{\frac{22}{15}}.$$

Hence, where the exponents of a quantity are fractional,

To Multiply, Rule.—*Add the exponents.*

2. Let it be required to multiply $a^{-\frac{3}{4}}$ by $a^{\frac{5}{6}}$.

Adding $-\frac{3}{4}$ and $\frac{5}{6}$, we have $\frac{1}{12}$. Hence, the product is $a^{\frac{1}{12}}$, or $\sqrt[12]{a}$.

213. By an explanation similar to that given in the preceding article, we derive the following rule. Where the exponents of a quantity are fractional,

To Divide, Rule.—*Subtract the exponent of the divisor from the exponent of the dividend.*

Perform the operations indicated in each of the following examples:

1. $a^{\frac{1}{2}}\times a^{\frac{2}{3}}$, and $a^{-\frac{1}{2}}\times a^{\frac{2}{3}}$. . . . . . Ans. $a^{\frac{7}{6}}$, and $a^{\frac{1}{6}}$.

2. $a^{\frac{3}{4}}c^{-1}\times a^2c^{\frac{3}{5}}$. . . . . Ans. $a^{\frac{11}{4}}c^{-\frac{2}{5}}$.

3. $\left(\frac{ay}{x}\right)^{\frac{1}{2}}\times\left(\frac{bx}{y^2}\right)^{\frac{1}{3}}\times\left(\frac{y^2}{a^3b^2}\right)^{\frac{1}{6}}$. . . . . . Ans. $\left(\frac{y}{x}\right)^{\frac{1}{6}}$.

4. $(a^{\frac{2}{3}}+a^{\frac{1}{3}}b^{\frac{1}{3}}+b^{\frac{2}{3}})(a^{\frac{1}{3}}-b^{\frac{1}{3}})$. . . . . Ans. $a-b$.

5. $(x^{\frac{1}{4}}y+y^{\frac{2}{3}})(x^{\frac{1}{4}}-y^{-\frac{1}{3}})$. . . . . . Ans. $x^{\frac{1}{2}}y-y^{\frac{1}{3}}$.

6. $(a+b)^{\frac{1}{m}}\times(a+b)^{\frac{1}{n}}\times(a-b)^{\frac{1}{m}}\times(a-b)^{\frac{1}{n}}$. Ans. $(a^2-b^2)^{\frac{m+n}{mn}}$.

7. $x^{\frac{2}{3}}\div x^{\frac{1}{4}}$, and $x^{\frac{3}{m}}y^n\div x^{\frac{2}{n}}y^m$ . . Ans. $x^{\frac{5}{12}}$, and $x^{\frac{3n-2m}{mn}}y^{n-m}$.

8. $(a^{\frac{3}{4}}-b^{\frac{3}{4}})\div(a^{\frac{1}{4}}-b^{\frac{1}{4}})$. . . . Ans. $a^{\frac{1}{2}}+a^{\frac{1}{4}}b^{\frac{1}{4}}+b^{\frac{1}{2}}$.

9. $(a-b^2)\div(a^{\frac{3}{4}}+a^{\frac{1}{2}}b^{\frac{1}{2}}+a^{\frac{1}{4}}b+b^{\frac{3}{2}})$. . . . . . Ans. $a^{\frac{1}{4}}-b^{\frac{1}{2}}$.

POWERS AND ROOTS OF QUANTITIES WITH FRACTIONAL EXPONENTS.

214. Since the m^{th} power of a quantity is the product of m factors, each equal to the quantity (Art. 172);

Therefore, to raise $a^{\frac{1}{n}}$ to the m^{th} power, we have

$$a^{\frac{1}{n}}\times a^{\frac{1}{n}}\times a^{\frac{1}{n}} \ldots \text{ to } m \text{ factors } = a^{\frac{m}{n}}.$$

Hence, to raise a quantity affected with a fractional exponent to any power,

Rule.—*Multiply the fractional exponent by the exponent of the power.*

$$\text{Thus, } (a^{\frac{1}{2}}b^{\frac{1}{3}})^4=a^{\frac{4}{2}}b^{\frac{4}{3}}=a^2b^{\frac{4}{3}}.$$

215. Conversely, to extract any root of a quantity affected by a fractional exponent,

Rule.—*Divide the exponent by the index of the root.*

Thus, $\sqrt[m]{a^{\frac{m}{n}}}=a^{\frac{m}{n}\div m}=a^{\frac{m}{n}\times\frac{1}{m}}=a^{\frac{1}{n}}$.

1. Raise $a^{\frac{1}{2}}b^{\frac{1}{3}}$ to the 4th power. . . . Ans. $a^2b^{\frac{4}{3}}$.

2. Raise $-2x^{\frac{1}{2}}y^{\frac{1}{3}}z^{\frac{1}{4}}$ to the 3d, 4th, and 6th powers.
Ans. $-8x^{\frac{3}{2}}yz^{\frac{3}{4}}$; $16x^2y^{\frac{4}{3}}z$; $64x^3y^2z^{\frac{3}{2}}$.

3. Find the square of $a-(ax-a^2)^{\frac{1}{2}}$.
Ans. $ax-2a(ax-a^2)^{\frac{1}{2}}$.

4. Find the cube of $a^{\frac{1}{3}}x^{-1}+a^{-\frac{1}{3}}x$.
Ans. $ax^{-3}+3a^{\frac{1}{3}}x^{-1}+3a^{-\frac{1}{3}}x+a^{-1}x^3$.

5. Find the cube roots of $(27a^3x)^{\frac{1}{2}}$ and $(27a^3x)^{\frac{1}{3}}$.
Ans. $3^{\frac{1}{2}}a^{\frac{1}{2}}x^{\frac{1}{6}}$, or $(3ax^{\frac{1}{3}})^{\frac{1}{2}}$; and $(3ax^{\frac{1}{3}})^{\frac{1}{3}}$.

6. Find the square root of $5x^3-4x(5cx)^{\frac{1}{2}}+4c$.
Ans. $5^{\frac{1}{2}}x^{\frac{3}{2}}-2c^{\frac{1}{2}}$.

7. Find the cube root of $\frac{1}{8}a^3-\frac{3}{2}a^2b^{\frac{1}{2}}+6ab-8b^{\frac{3}{2}}$.
Ans. $\frac{1}{2}a-2b^{\frac{1}{2}}$.

VII. EQUATIONS CONTAINING RADICALS.

216. In the solution of questions containing radicals, the method to be pursued will often depend on the judgment of the pupil, as many of them can be solved in different ways, and the shortest processes can only be learned from *practice.*

1st. When the equation to be solved contains only one radical expression, transpose it to one side of the equation and the rational terms to the other; then involve both sides to a power corresponding to the radical sign.

1. Given, $\sqrt[3]{(a^3+x)}-a=c$, to find x.

Transposing, $\sqrt[3]{(a^3+x)}=c+a$;
Cubing, $a^3+x=c^3+3ac^2+3a^2c+a^3$;
Whence, $x=c^3+3ac^2+3a^2c$.

2d. When a radical expression occurs under the radical sign, the operation of involution must be repeated.

2. Given $\sqrt{x-\sqrt{1-x}}=1-\sqrt{x}$, to find x.

Squaring, $x-\sqrt{1-x}=1-2\sqrt{x}+x$;

Canceling x on each side, and squaring again,

$$1-x=1-4\sqrt{x}+4x.$$

Canceling 1 on each side, transposing, squaring, and reducing,

We find, $x=\frac{16}{25}$.

3d. When there are two or more radical expressions, it is generally preferable to make one of them stand alone before performing the process of involution.

3. Given, $\sqrt{x+9}-\sqrt{x}=1$, to find x.

Transposing, $-\sqrt{x}$, we have $\sqrt{x+9}=1+\sqrt{x}$.
Squaring each side, $x+9=1+2\sqrt{x}+x$;

Canceling x on each side, transposing, and dividing by 2,

$$\sqrt{x}=4; \text{ hence, } x=16.$$

In some cases, however, it is preferable, when an equation contains two radical expressions, to retain them both on the same side. Thus, the following equation will be cleared of radicals at once, by squaring each side:

$$\sqrt{\left(\frac{x+a}{x-a}\right)}+\sqrt{\left(\frac{x-a}{x+a}\right)}=b. \qquad \text{Ans. } x=\frac{ab}{\sqrt{b^2-4}}.$$

4. $\sqrt{(x+5)}+3=8-\sqrt{x}$. . . . . Ans. $x=4$.

5. $\sqrt{1+\sqrt{(3+\sqrt{6x})}}=2$. . . . . Ans. $x=6$.

6. $\sqrt{x+a}=\sqrt{x}+a$. . . . . Ans. $x=\frac{(a-1)^2}{4}$.

7. $\sqrt{2x-3a}+\sqrt{2x}=3\sqrt{a}$. . . . . Ans. $x=2a$.

8. $\sqrt{\{13+\sqrt{[7+\sqrt{(3+\sqrt{x})}]}\}}=4$. . . . . Ans. $x=1$.

9. $\sqrt{2+x}+\sqrt{x}=\dfrac{4}{\sqrt{2+x}}$. . . . . Ans. $x=\frac{2}{3}$.

10. $\sqrt{a+x}+\sqrt{\dfrac{a}{x}}=\sqrt{x}$. . . . . Ans. $x=\dfrac{a}{a+2\sqrt{a}}$.

11. $\sqrt{x+13}-\sqrt{x-11}=2$. . . . . Ans. $x=36$.

12. $a\sqrt{x}+b\sqrt{x}-c\sqrt{x}=d$. . . . . Ans $x=\dfrac{d^2}{(a+b-c)^2}$.

13. $\dfrac{x-ax}{\sqrt{x}}=\dfrac{\sqrt{x}}{x}$. . . . . Ans. $x=\dfrac{1}{1-a}$.

14. $x+a=\sqrt{a^2+x\sqrt{(b^2+x^2)}}$. . . . . Ans. $x=\dfrac{b^2-4a^2}{4a}$.

15. $\dfrac{x-4}{\sqrt{x}+2}=5\sqrt{x}-8+\dfrac{3\sqrt{x}}{2}$. . . . . Ans. $x=\frac{144}{121}$.

16. $\dfrac{x-a}{\sqrt{x}+\sqrt{a}}=\dfrac{\sqrt{x}-\sqrt{a}}{3}+2\sqrt{a}$. . . . . . Ans. $x=16a$.

17. $\dfrac{3x-1}{\sqrt{3x}+1}=1+\dfrac{\sqrt{3x}-1}{2}$. . . . . Ans. $x=3$.

18. $\sqrt{4a+x}=2\sqrt{b+x}-\sqrt{x}$. . . . . . Ans. $x=\dfrac{(b-a)^2}{2a-b}$.

19. $\sqrt{\dfrac{b}{a+x}}+\sqrt{\dfrac{c}{a-x}}=\sqrt[4]{\dfrac{4bc}{a^2-x^2}}$. . . . . Ans. $x=\dfrac{a(b+c)}{b-c}$.

20. $\dfrac{\sqrt{x+a}+\sqrt{x}}{\sqrt{x+a}-\sqrt{x}}=c$. . . . . Ans. $x=\dfrac{a(c-1)^2}{4c}$.

21. $\sqrt{\sqrt{x}+3}-\sqrt{\sqrt{x}-3}=\sqrt{2\sqrt{x}}$. . . . . . Ans. $x=9$.

22. $\dfrac{1}{x}+\dfrac{1}{a}=\sqrt{\left\{\dfrac{1}{a^2}+\sqrt{\left(\dfrac{1}{b^2x^2}+\dfrac{1}{x^4}\right)}\right\}}$. Ans. $x=\dfrac{4ab^2}{a^2-4b^2}$.

23. $\sqrt{(1+a)^2+(1-a)x}+\sqrt{(1-a)^2+(1+a)x}=2a$. Ans. $x=8$.

VIII. INEQUALITIES.

217. In the discussion of problems, it often becomes necessary to compare quantities that are *unequal*, and to operate upon them so as to determine the values of the unknown quantities, or to establish certain relations between them.

In most cases the methods of operating on equations apply to inequalities, but there are some exceptions.

218. In the theory of inequalities, it is convenient to consider negative quantities less than zero.

In comparing two negative quantities, that is considered the least which contains the greatest number of units; thus, $0>-1$, and $-3>-5$.

Two inequalities are said to subsist in the *same* sense, when the greater quantity stands on the right in both, or on the left in both; as, $5>3$ and $7>4$.

Two inequalities are said to subsist in a *contrary* sense, when the greater stands on the *right* in one and on the *left* in the other; as, $5>1$ and $4<8$.

219. Proposition I.—*If the same quantity, or equal quantities, be added to or subtracted from both members of an inequality, the resulting inequality will continue in the same sense.*

Thus, ... $7>5$.
Adding 4 to each member, . . $11>9$.
Subtracting 4 from each member, . $3>1$.

Also, $-5<-3$; and by adding and subtracting 4,

$$-1<+1, \text{ and } -9<-7.$$

Similarly, if $a>b$, then $a+c>b+c$, or $a-c>b-c$. Hence,

Any quantity may be transposed from one side of an inequality to the other, if at the same time its sign be changed.

220. Proposition II.—*If two inequalities exist in the same sense, the corresponding members may be added together, and the resulting inequality will exist in the same sense.*

Thus, if $7>6$, and $5>4$; then,
$$7+5>6+4, \text{ or } 12>10.$$

When two inequalities exist in the same sense, if we *subtract* the corresponding members, the resulting in-

equality will exist, sometimes in the *same*, and sometimes in a *contrary* sense.

First, $7>3$ By subtracting, we find the resulting inequality
$\quad 4>1$ exists in the *same* sense.

$\overline{\quad 3>2}$

Second, $10>9$ In this case, after subtracting, we find the
$\quad 8>3$ resulting inequality exists in a *contrary*
$\overline{\quad 2<6}$ sense.

In general, if $a>b$ and $c>d$, then, according to the particular values of a, b, c, and d, we may have $a-c>b-d$, $a-c<b-d$, or $a-c=b-d$.

221. Proposition III.—*If the two members of an inequality be multiplied or divided by a positive number, the resulting inequality will exist in the same sense.*

Thus, $8>4$ and $8\times3>4\times3$, or $24>12$.
Also, $8\div2>4\div2$, or $4>2$.

This principle enables us to clear an inequality of fractions.

If the multiplier be a negative number, the resulting inequality will exist in a contrary sense.

Thus, $-3<-1$, but $-3\times-2>-1\times-2$, or $6>2$.

From this principle we derive

222. Proposition IV.—*The signs of all the terms of both members of an inequality may be changed, if at the same time we establish the resulting inequality in a contrary sense.*

For this is the same as multiplying both members by -1.

223. Proposition V.—*Both members of a positive inequality may be raised to the same power, or have the same root extracted, and the resulting inequality will exist in the same sense.*

Thus, $2<3$ and $2^2<3^2$, $2^3<3^3$; or $4<9$, $8<27$; and so on.

Also, $25>16$, and $\sqrt{25}>\sqrt{16}$, or $5>4$; and so on.

But if the signs of both members of an inequality are not positive, the resulting inequality may exist in the same, or in a contrary sense.

Thus, $3>-2$, and $3^2>(-2)^2$, or $9>4$.
But, $-3<-2$, and $(-3)^2>(-2)^2$, or $9>4$.

EXAMPLES INVOLVING THE PRINCIPLES OF INEQUALITIES.

1. Five times a certain whole number increased by 4, is greater than twice the number increased by 19; and 5 times the number diminished by 4, is less than 4 times the number increased by 4. Required the number.

Let $x=$ the number.
Then, $5x+4>2x+19$, (1)
$5x-4<4x+4$. (2)
$5x-2x>19-4$, from eq. (1) by transposing,
$3x>15$, by reducing,
$x>5$, by dividing both members by 3.
$5x-4x<4+4$, from eq. (2) by transposing,
$x<8$, by reducing.

Hence, the number is greater than 5 and less than 8, consequently either 6 or 7 will fulfill the conditions.

2. If $4x-7<2x+3$, and $3x+1>13-x$, find x. Ans. $x=4$.

3. Find the limit of x in $7x-3>32$. Ans. $x>5$.

4. Of x in the inequality $5+\frac{1}{3}x<8+\frac{1}{4}x$ Ans. $x<36$.

5. Show that $\frac{a+c+e}{b+d+f}>$ the least, and $<$ the greatest of the fractions, $\frac{a}{b}$, $\frac{c}{d}$, $\frac{e}{f}$, each letter representing a positive quantity.

Suppose $\frac{a}{b}$ to be the greatest, and $\frac{c}{d}$ the least, of the fractions, $\frac{a}{b}$, $\frac{c}{d}$, $\frac{e}{f}$. Then, $\frac{a}{b}>\frac{c}{d}$, $\frac{c}{d}=\frac{c}{d}$, $\frac{e}{f}>\frac{c}{d}$; and $\frac{a}{b}=\frac{a}{b}$, $\frac{c}{d}<\frac{a}{b}$, $\frac{e}{f}<\frac{a}{b}$.

$$a > \frac{bc}{d},\quad c = \frac{cd}{d},\quad e > \frac{cf}{d}.\quad \text{(Art. 221.)}$$

$$a = \frac{ab}{b},\quad c < \frac{ad}{b},\quad e < \frac{af}{b}.\quad \text{(Art. 221.)}$$

$$a+c+e > (b+d+f)\frac{c}{d}.\quad \text{(Arts. 219, 220.)}$$

$$a+c+e < (b+d+f)\frac{a}{b}.\quad \text{(Arts. 219, 220.)}$$

Hence, $\dfrac{a+c+e}{b+d+f} > \dfrac{c}{d}$; and $\dfrac{a+c+e}{b+d+f} < \dfrac{a}{b}$.

6. It is required to prove that the sum of the squares of any two *unequal* magnitudes is always greater than twice their product.

Since the square of every quantity, whether positive or negative, is positive, it follows that

$(a-b)^2$, or $a^2-2ab+b^2>0$.

Adding, $+2ab$ to each side (Art. 219),

$a^2+b^2>2ab$, which was required to be proved.

Most of the inequalities usually met with, are made to depend ultimately upon this principle.

7. Which is greater, $\sqrt{5}+\sqrt{14}$ or $\sqrt{3}+3\sqrt{2}$?

Ans. the former.

8. Given $\frac{1}{4}(x+2)+\frac{1}{3}x<\frac{1}{2}(x-4)+3$ and $>\frac{1}{2}(x+1)+\frac{1}{3}$, to find x. Ans. $x=5$.

9. The double of a certain number increased by 7, is not greater than 19, and its triple diminished by 5, is not less than 13. Required the number. Ans. 6.

10. Show that every fraction + the fraction inverted, is greater than 2; that is, that $\frac{a}{b}+\frac{b}{a}>2$.

11. Show that $a^2+b^2+c^2>ab+ac+bc$, unless $a=b=c$.

12. If $x^2=a^2+b^2$, and $y^2=c^2+d^2$, which is greater, xy or $ac+bd$? Ans. xy.

13. Show that $abc>(a+b-c)(a+c-b)(b+c-a)$, unless $a=b=c$.

VII. QUADRATIC EQUATIONS.

224. A **Quadratic Equation,** or an equation of the *second degree*, is one in which the greatest exponent of the unknown quantity is 2; as, $x^2+x=a$.

An equation containing two or more unknown quantities, in which the greatest sum of the exponents of the unknown quantities in one term is 2, is also a Quadratic Equation; as, $xy=a$, $xy-x-y=c$.

225. Quadratic equations, containing only one unknown quantity, are divided into two classes, *pure* and *affected*.

A **Pure Quadratic Equation** is one that contains only the second power of the unknown quantity, and known terms; as,

$$x^2+2=47-4x^2, \text{ and } ax^2+b=cx^2-d.$$

A pure quadratic equation is also called *an incomplete equation of the second degree.*

An **Affected Quadratic Equation** is one that contains both the first and second power of the unknown quantity, and known terms; as,

$$5x^2+7x=34, \text{ and } ax^2-bx^2+cx-dx=e-f.$$

An affected quadratic equation is also called *a complete equation of the second degree.*

226. The general form of a pure equation is $ax^2=b$.

The general form of an affected equation is $ax^2+bx=c$.

Every quadratic equation containing only one unknown quantity may be reduced to one of these forms.

For, in a pure equation, all the terms containing x^2 may be collected into one term of the form, ax^2; and all the known quantities into another, as b.

So, in an affected equation, all the terms containing x^2 may be reduced to one term, as ax^2; and those containing x to one, as bx; and the known terms to one, as c.

PURE QUADRATIC EQUATIONS.

227.—1. Let it be required to find the value of x in the equation, $\frac{1}{3}x^2-3+\frac{5}{12}x^2=12\frac{3}{4}-x^2$.

Clearing of fractions, $4x^2-36+5x^2=153-12x^2$;
Transposing and reducing, $21x^2=189$;
Dividing, $x^2=9$;

Extracting the square root of both members,

$$x=\pm 3;\text{ that is, } x=+3, \text{ or } x=-3.$$

Verification. $\frac{1}{3}(+3)^2-3+\frac{5}{12}(+3)^2=12\frac{3}{4}-(+3)^2$.
$3-3+3\frac{3}{4}=12\frac{3}{4}-9$; or $3\frac{3}{4}=3\frac{3}{4}$.

Since the square of -3 is the same as the square of $+3$, the value $x=-3$, will give the same result as $x=+3$.

2. Given $ax^2+b=d+cx^2$, to find the value of x.

Transposing, . . $ax^2-cx^2=d-b$;
Factoring, ... $(a-c)x^2=d-b$;
Dividing, ... $x^2=\dfrac{d-b}{a-c}$;

$$x=\pm\sqrt{\frac{d-b}{a-c}}.$$

From the preceding examples, we derive the following

Rule for the Solution of a Pure Equation.—*Reduce the equation to the form* ax²=b. *Divide by the coëfficient of* x², *and extract the square root of both members.*

228. If we solve the equation $ax^2=b$, we have,

$$x=\pm\sqrt{\frac{b}{a}};\text{ that is, } x=+\sqrt{\frac{b}{a}}, \text{ and } x=-\sqrt{\frac{b}{a}}.$$

The equation may be verified by substituting either of these values of x. Hence, we infer,

1st. *That in every pure equation the unknown quantity has two values, or roots, and only two.*

2d. *That these roots are equal in value, but have contrary signs.*

1. $11x^2-44=5x^2+10$. . . . . . . Ans. $x=\pm3$.
2. $\frac{1}{3}(x^2-12)=\frac{1}{4}x^2-1$. . . . . . . Ans. $x=\pm6$.
3. $(x+2)^2=4x+5$. . . . . Ans. $x=\pm1$
4. $\dfrac{8}{1-2x}+\dfrac{8}{1+2x}=25$. . . . . . . Ans. $x=\pm.3$.
5. $\dfrac{x+7}{x^2-7x}-\dfrac{x-7}{x^2+7x}=\dfrac{7}{x^2-73}$. . . . Ans. $x=\pm9$.
6. $\dfrac{a}{b+x}+\dfrac{a}{b-x}=c$. . . . Ans. $x=\pm\dfrac{1}{c}\sqrt{b^2c^2-2abc}$.
7. $x\sqrt{6+x^2}=1+x^2$. . . . . Ans. $x=\pm\frac{1}{2}$.
8. $x+\sqrt{a^2+x^2}=\dfrac{2a^2}{\sqrt{a^2+x^2}}$. . . . Ans. $x=\pm\dfrac{a}{3}\sqrt{3}$.
9. $\dfrac{2}{x+\sqrt{2-x^2}}+\dfrac{2}{x-\sqrt{2-x^2}}=x$. . Ans. $x=\pm\sqrt{3}$.
10. $\dfrac{a-\sqrt{a^2-x^2}}{a+\sqrt{a^2-x^2}}=b$. . . . . . Ans. $x=\pm\dfrac{2a\sqrt{b}}{b+1}$.

QUESTIONS PRODUCING PURE EQUATIONS.

229. For the statement of the equation, see Art. 154.

1. What two numbers have the ratio of 2 to 5, the sum of whose squares is 261?

Let $2x$ and $5x=$ the numbers.
Then, $4x^2+25x^2=29x^2=261$;
Whence, $x^2=9$, and $x=3$.
Hence, $2x=6$, and $5x=15$ the required numbers.

2. The square of a certain number diminished by 17, is equal to 130 diminished by twice the square of the number. Required the number. Ans. 7.

3. Required a certain number, which being subtracted from 10 and the remainder multiplied by the number itself, gives the same product as 10 times the remainder after subtracting $6\frac{2}{5}$ from the number. Ans. 8.

4. What number is that, the $\frac{1}{3}$ part of whose square being subtracted from 30, leaves a remainder equal to $\frac{1}{4}$ of its square increased by 9? Ans. 6.

5. There are two numbers whose difference is $\frac{2}{9}$ of the greater, and the difference of their squares is 128; find them. Ans. 18 and 14.

6. Divide 21 into two such parts, that the square of the less shall be to that of the greater as 4 to 25.

Let x and $21-x=$ the parts.
Then, $x^2 : (21-x)^2 : : 4 : 25$;
Or, (Arith., Art. 200,) $25x^2=4(21-x)^2$;
Extracting square root, $5x=2(21-x)$;
Whence, $x=6$, and $21-x=15$.

7. Divide 14 into two such parts, that the quotient of the greater divided by the less, shall be to the quotient of the less by the greater, as 16 to 9. Ans. 6 and 8.

8. What number is that which being added to 20 and subtracted from 20, the product of the sum and difference shall be 319? Ans. 9.

9. Find two numbers, whose product is 126, and the quotient of the greater by the less $3\frac{1}{2}$. Ans. 6 and 21.

10. The product of two numbers is p, and their quotient q. Required the numbers. Ans. $\sqrt{pq}$ and $\sqrt{\frac{p}{q}}$.

11. The sum of the squares of two numbers is 370, and the difference of their squares 208. Required the numbers. Ans. 9 and 17.

12. The sum of the squares of two numbers is c, and the difference of their squares d. Required the numbers.

Ans. $\frac{1}{2}\sqrt{2(c+d)}$, and $\frac{1}{2}\sqrt{2(c-d)}$.

13. A certain sum of money is lent at 5 % per annum. If we multiply the number of dollars in the principal by the number of dollars in the interest for 3 mon., the product is 720. What is the sum lent? Ans. \$240.

14. It is required to find 3 numbers, such that the product of the 1st and 2d $=a$, the product of the 1st and 3d $=b$, and the sum of the squares of the 2d and 3d $=c$.

Ans. $\sqrt{\left(\frac{a^2+b^2}{c}\right)}$, $a\sqrt{\left(\frac{c}{a^2+b^2}\right)}$, and $b\sqrt{\left(\frac{c}{a^2+b^2}\right)}$.

15. The spaces through which a body falls in different periods of time, being to each other as the squares of those times, in how many sec. will a body fall through 400 ft., the space it falls through in one sec. being 16.1 ft.?

Let $x=$ the required number of seconds.
Then, $16.1 : 400 :: 1^2 : x^2$; whence, $x=4.98+$ sec.

In what time will it fall 1000 ft.? Ans. 7.88+ sec.

16. What two numbers are as 3 to 5, and the sum of whose cubes is 1216?

Let $3x$ and $5x=$ the numbers;
Then, $27x^3+125x^3=152x^3=1216$;
Whence, $x^3=8$, and $x=\sqrt[3]{8}=2$.
Hence, the numbers are 6 and 10.

This is properly a pure equation of the *third* degree; but questions producing such equations are generally arranged with those of the second degree.

17. A money safe contains a certain number of drawers. In each drawer there are as many divisions as there are drawers, and in each division there are four times as many dollars as there are drawers. The whole sum in the safe is \$5324; what is the number of drawers? Ans. 11.

18. A and B set out to meet each other; A leaving the town C at the same time that B left D. They traveled the direct road from C to D, and on meeting, it appeared that A had traveled 18 miles more than B; and that A could have gone B's journey in $15\frac{3}{4}$ days, but B would have been 28 days in performing A's journey. What is the distance between C and D? Ans. 126 miles.

19. Two men, A and B, engaged to work for a certain number of days at different rates. At the end of the time, A, who had played 4 days, received 75 shillings; but B, who had played 7 days, received only 48 shillings. Had B played only 4 days, and A 7 days, they would have received the same sum. For how many days were they engaged? Ans. 19.

20. A vintner draws a certain quantity of wine out of a full vessel that holds 256 gal.; and then filling the vessel with water, draws off the same number of gal. as before, and so on for four draughts, when there were only 81 gal. of pure wine left. How much wine did he draw each time?

Ans. 64, 48, 36, and 27 gal.

AFFECTED QUADRATIC EQUATIONS.

230.—1. Required to find the value of x in the equation,

$$x^2-6x+9=4.$$

It is evident, from Art. 184, that the first member of this equation is a perfect square. By extracting the square root of both members,

We find, ... $x-3=\pm 2$;

Whence, ... $x=3\pm 2=3+2=5$, or $3-2=1$.

Verification. $(5)^2-6(5)+9=4$; that is, $25-30+9=4$.

$(1)^2-6(1)+9=4$; that is, $1-6+9=4$.

Hence, x has *two values*, $+5$, and $+1$, either of which verifies the equation.

2. Required to find the value of x in the equation,

$$x^2-6x=27.$$

As the left member of this equation is not a perfect square, we can not find the value of x by extracting the square root, as in the preceding example. We may, however, render the first member a perfect square by adding 9 to it.

This may be done provided the same number be added to the other member, to preserve the equality. The equation then becomes,

$$x^2-6x+9=36.$$

Extracting the square root, $x-3=\pm6$.

Whence, $x=3\pm6=+9$, or -3, either of which values of x will verify the equation.

231. We will now proceed to explain the method of completing the square.

Since every affected equation (Art. 226) may be reduced to the form,

$$ax^2+bx=c,$$

Dividing both sides by a, $x^2+\frac{b}{a}x=\frac{c}{a}$.

For the sake of simplicity, let $\frac{b}{a}=2p$, and $\frac{c}{a}=q$. As each of these fractions may be either positive or negative, the equation must assume one of the four following forms:

$$x^2+2px=q. \quad (1)$$
$$x^2-2px=q. \quad (2)$$
$$x^2+2px=-q. \quad (3)$$
$$x^2-2px=-q. \quad (4)$$

Hence,

Every affected equation may be reduced to the form $x^2\pm2px=\pm q$.

It will now be shown that the first member of this equation may always be made a perfect square.

We may consider x^2+2px as the first two terms of the square of a binomial, the third term being unknown or lost.

Extracting the root of x^2, we find that the first term of the binomial must be x. We next observe that $2px$ is *twice* the product of the first term by the second; therefore, p, which is *half the coëfficient of* x, is the second term of the binomial, and its square, p^2,

added to x^2+2px, will render it a perfect square. But, to preserve the equality, we must add the same quantity to both sides.

This gives, . . $x^2+2px+p^2=q+p^2$;
Extracting the square root, $x+p=\pm\sqrt{q+p^2}$;
Transposing, ... $x=-p\pm\sqrt{q+p^2}$.

It is obvious that in each of the remaining three forms, the square may be completed on the same principle.

Solving equations (2), (3), and (4), and collecting together the four different forms, we have the following table:

(1)	$x^2+2px=q.$	$x=-p\pm\sqrt{q+p^2}.$
(2)	$x^2-2px=q.$	$x=+p\pm\sqrt{q+p^2}.$
(3)	$x^2+2px=-q.$	$x=-p\pm\sqrt{-q+p^2}.$
(4)	$x^2-2px=-q.$	$x=+p\pm\sqrt{-q+p^2}.$

From the preceding we derive the following

Rule for the Solution of an Affected Equation.—1st. *Reduce the equation, by clearing of fractions and transposition, to the form* $ax^2+bx=c$.

2d. *If the coëfficient of* x^2 *is minus, change the signs of all the terms, or multiply each term by* —1.

3d. *Divide each side of the equation by the coëfficient of* x^2

4th. *Add to each member the square of half the coëfficient of* x.

5th. *Extract the square root of both sides, transpose the known term to the second member, and find the value of* x.

REMARK.—Although from the equation $x^2=m^2$, we have $\pm x=\pm m$; that is, $+x=+m(1)$, $+x=-m(2)$, $-x=+m(3)$, and $-x=-m(4)$, it is evident that equations (1) and (4) are the same equation, as also (2) and (3). Hence, $+x=\pm m$, embraces all the values of x. For the same reason it is necessary to take only the plus sign of the square root of $(x+p)^2$.

1. Given $17x-2x^2=32-3x$, to find x.

Transposing, ... $-2x^2+20x=32$;
Changing signs, and reducing, $x^2-10x=-16$;
Completing the square by adding $(\frac{10}{2})^2=25$ to both sides,

$$x^2-10x+25=-16+25=9;$$

Extracting the root, $x-5=\pm3$;
Whence, . . $x=5\pm3=8$, or 2.

Verification. $17(8)-2(8)^2=32-3(8)$, or $+8=+8$.
$17(2)-2(2)^2=32-3(2)$, or $+26=+26$.

2. Given $3x^2-2x=65$, to find x.

Dividing by 3, . $x^2-\frac{2}{3}x=\frac{65}{3}$;
Completing the square, $x^2-\frac{2}{3}x+(\frac{1}{3})^2=\frac{65}{3}+(\frac{1}{3})^2=\frac{196}{9}$.
Extracting the root, $x-\frac{1}{3}=\pm\frac{14}{3}$.
Whence, . $x=\frac{1}{3}\pm\frac{14}{3}=5$, or $-4\frac{1}{3}$.

Both of which values verify the equation.

3. Given $4a^2-2x^2+2ax=18ab-18b^2$, to find x.

Transposing, . . $-2x^2+2ax=-4a^2+18ab-18b^2$;
Dividing by -2, . $x^2-ax=2a^2-9ab+9b^2$;
Completing the square, $x^2-ax+\dfrac{a^2}{4}=\dfrac{9a^2}{4}-9ab+9b^2$;
Extracting root, . . $x-\dfrac{a}{2}=\pm\left(\dfrac{3a}{2}-3b\right)$;
Whence, $x=\dfrac{a}{2}+\left(\dfrac{3a}{2}-3b\right)=2a-3b$, or $-a+3b$.

4. Given $x+\sqrt{(5x+10)}=8$, to find x.

By transposition, $\sqrt{(5x+10)}=8-x$;
By squaring, . . . $5x+10=64-16x+x^2$;
Or, $x^2-21x=-54$;
Completing the square, $x^2-21x+(\frac{21}{2})^2=\frac{441}{4}-54=\frac{225}{4}$;
Extracting the root, $x-\frac{21}{2}=\pm\frac{15}{2}$;
Whence, $x=\frac{21}{2}\pm\frac{15}{2}=\frac{36}{2}=18$, or $\frac{6}{2}=3$.

These two values of x are the roots of the equation, $x^2-21x=-54$, but they will not both verify the original equation.

For, the proposed equation might have been $x\pm\sqrt{(5x+10)}=8$; and the operations which have been employed would result in the same equation, $x^2-21x=-54$, whether the sign of the radical part be $+$ or $-$.

Hence, in the equation $x+\sqrt{(5x+10)}=8$, the value of x is 3; but in the equation $x-\sqrt{(5x+10)}=8$, the value is 18.

5. $x^2+4x=60$. . . . . . . Ans. $x=6$, or -10.

6. $x^2-4x=60$. . . . . . . Ans. $x=10$, or -6.

7. $x^2+16x=-60$. . . . Ans. $x=-6$, or -10.

8. $x^2-16x=-60$. . . . . . Ans. $x=6$, or 10.

9. $x^2-6x=6x+28$. . . . Ans. $x=14$, or -2.

10. $\frac{x^2}{10}+350-12x=0$. . . . Ans. $x=70$, or 50.

11. $2x=4+\frac{6}{x}$. . . . . Ans. $x=3$, or -1.

12. $3x^2+10x=57$. . . . . . Ans. $x=3$, or $-6\frac{1}{3}$.

13. $(x-1)(x-2)=1$. . . . . Ans. $x=\frac{1}{2}(3\pm\sqrt{5})$.

14. $\frac{1}{2}x^2-\frac{1}{4}x+2=9$. . . . . Ans. $x=4$, or $-3\frac{1}{2}$.

15. $x=1+\frac{110}{x}$. . . . . . . Ans. $x=11$, or -10.

16. $\frac{x+22}{3}-\frac{4}{x}=\frac{9x-6}{2}$. . . . . Ans. $x=2$, or $\frac{12}{25}$.

17. $\frac{2x^2}{3}+3\frac{1}{2}=\frac{x}{2}+8$. . . . . Ans. $x=3$, or $-2\frac{1}{4}$.

18. $17x^2+19x=1848$. . . Ans. $x=9\frac{15}{17}$, or -11.

19. $\frac{z^2}{3}-\frac{z}{10}+\frac{1}{6}=\frac{1}{5}$. . . . . . Ans. $z=\frac{1}{2}$, or $-\frac{1}{5}$.

20. $3x-\frac{1}{4}x^2=10$. . . . . . Ans. $x=6\pm2\sqrt{-1}$.

21. $\frac{1}{x^2-3x}+\frac{1}{x^2+4x}=\frac{9}{8x}$. . . Ans. $x=4$, or $-3\frac{2}{9}$.

22. $\frac{x+4}{3}-\frac{7-x}{x-3}=\frac{4x+7}{9}-1$. Ans. $x=21$, or 5.

23. $x+\frac{1}{x}=\frac{4}{\sqrt{3}}$. . . . . Ans. $x=\sqrt{3}$, or $\frac{1}{3}\sqrt{3}$.

24. $\dfrac{x+\frac{1}{x}}{x-\frac{1}{x}}+\dfrac{1+\frac{1}{x}}{1-\frac{1}{x}}=\frac{13}{4}$. . . . . Ans. $x=3$, or $-\frac{7}{5}$.

25. $\frac{x}{a}+\frac{a}{x}-\frac{2}{a}=0$. . . . . Ans. $x=1\pm\sqrt{(1-a^2)}$.

26. $\frac{a^2-b^2}{c}=2ax-cx^2$. . . . . . . Ans. $x=\frac{a\pm b}{c}$.

27. $x^2-(a+b)x+ab=0$. . . . . Ans. $x=a$, or b.

28. $(a-b)x^2-(a+b)x+2b=0$. Ans. $x=1$, or $\frac{2b}{a-b}$.

29. $mqx^2-mnx+pqx-np=0$. Ans. $x=\frac{n}{q}$, or $-\frac{p}{m}$.

30. $\frac{x^2}{a^{\frac{1}{2}}+b^{\frac{1}{2}}}-(a^{\frac{1}{2}}-b^{\frac{1}{2}})x=\frac{1}{(ab^2)^{-\frac{1}{2}}+(a^2b)^{-\frac{1}{2}}}$.

Ans. $x=a$, or $-b$.

31. $adx-acx^2=bcx-bd$. . Ans. $x=\frac{d}{c}$, or $-\frac{b}{a}$.

32. $\sqrt{(x+5)}=\frac{12}{\sqrt{(x+12)}}$. . Ans. $x=4$, or -21.

33. $\frac{\sqrt{4x}+2}{4+\sqrt{x}}=\frac{4-\sqrt{x}}{\sqrt{x}}$. . . . . Ans. $x=4$.

34. $\sqrt{x^3}-2\sqrt{x}=x$. . . . Ans. $x=4$.

35. $\sqrt{x+a}-\sqrt{x+b}=\sqrt{2x}$.

Ans. $x=-\frac{a+b}{2}\pm\frac{1}{2}\sqrt{2a^2+2b^2}$.

36. $\sqrt{a+x}+\sqrt{a-x}=\frac{12a}{5\sqrt{a+x}}$. Ans. $x=\frac{4a}{5}$, or $\frac{3a}{5}$.

232. Hindoo Method of Solving Quadratics.—When an equation is brought to the form $ax^2+bx=c$, it may be reduced to a simple equation, without dividing by the coëfficient of x^2, thus avoiding fractions.

If we multiply every term of the equation $ax^2+bx=c$, by *four* times the coëfficient of the *first* term, and add to both sides the *square* of the coëfficient of the *second* term, we shall have

$$4a^2x^2+4abx+b^2=4ac+b^2.$$

Now, the first member of this equation is a perfect square, and by extracting the square root of both sides, we have

$2ax+b=\pm\sqrt{4ac+b^2}$, which is a simple equation. Hence, the

Hindoo Rule for the Solution of Quadratic Equations.—1st. *Reduce the equation to the form* $ax^2+bx=c$.

2d. *Multiply both sides by four times the coëfficient of* x^2.

3d. *Add the square of the coëfficient of* x *to each side, extract the square root, and finish the solution.*

1. Given $2x^2-5x=3$, to find x.

Multiplying both sides by 8, *four* times the coëfficient of x^2,

We have . . . . $16x^2-40x=24$.

Adding to each side 25, which is the square of the coëfficient of x,

We have . $16x^2-40x+25=49$;

Extracting the root, $4x-5=\pm 7$; whence, $x=3$, or $-\frac{1}{2}$.

Find the value of the unknown quantity in each of the following examples by the *Hindoo* Rule:

2. $3x^2+5x=2$. . . . . Ans. $x=\frac{1}{3}$, or -2.
3. $x^2+x=30$. . . . . Ans. $x=5$, or -6.
4. $x^2-x=72$. . . . . Ans. $x=9$, or -8.
5. $\dfrac{40}{x-5}+\dfrac{27}{x}=13$. . . . . . Ans. $x=9$, or $\frac{15}{13}$.

By an inspection of the forms given in Art. 231, it will be seen that the value of the unknown quantity may be found without the formality of completing the square, by the following

Rule.—*Reduce the equation to the form* $x^2+2px=q$. *The unknown quantity will then be equal to one half the coëfficient of its first power taken with a contrary sign, plus or minus the square root of the square of the number last written together with the known quantity in the second member of the equation taken with its proper sign.*

Thus, let $x^2+16x=-60$.

Then, $x=-8\pm\sqrt{64-60}=-8\pm\sqrt{4}=-8\pm 2$.

$x=-6$, or -10.

After some exercise in completing the square, it is best to employ this last method.

PROBLEMS PRODUCING AFFECTED EQUATIONS.

233.—1. A person bought a certain number of sheep for \$40, and if he had bought 2 more for the same sum they would have cost \$1 apiece less. Required the number of sheep, and the price of each.

Let $x=$ the number of sheep.

Then, $\frac{40}{x}=$ the price of one.

And $\frac{40}{x+2}=$ the price of one, on the second supposition.

Therefore, $\frac{40}{x+2}=\frac{40}{x}-1$, by the question.

Solving this eq., $x=-1\pm 9=8$, or -10, number of sheep.

And $\frac{40}{x}=\frac{40}{8}=\$5$, price of each. Also, $\frac{40}{x}=\frac{40}{-10}=-4$.

The negative value, —10, to fulfill the conditions of the question in an arithmetical sense, must be modified, on the principles explained in Art. 164, thus:

A person *sells* a certain number of sheep for \$40. If he had sold 2 *fewer* for the same sum he would have received \$1 apiece *more* for them. Required the number sold.

2. Find a number such, that if 17 times the number be diminished by its square, the remainder shall be 70.

Let $x=$ the number.
Then, $17x-x^2=70$.
Or, $x^2-17x=-70$.
Whence, $x=7$, or 10.

In this case, *both* values of x satisfy the question in its arithmetical sense.

Thus, $17\times 7-7^2=119-49=70$.
Or, $17\times 10-10^2=170-100=70$.

3. Of a number of bees, after $\frac{8}{9}$, and the square root of $\frac{1}{2}$ of them, had flown away, there were two remaining; what was the number at first?

To avoid radicals, let $2x^2$ represent the number of bees at first;

$$\text{Then, } \frac{16x^2}{9}+x+2=2x^2.$$

Whence, $x=6$, or $-1\frac{1}{2}$; but the latter value, being *fractional*, though satisfying the equation, is excluded by the nature of the question; the number of bees is $2\times6^2=72$.

4. Divide a into two parts, whose product shall be b^2.

Let $x=$ one part; then, $a-x=$ the other.

Therefore, $x(a-x)$, or $ax-x^2=b^2$.

Whence, $x=\frac{1}{2}(a\pm\sqrt{a^2-4b^2})$; that is,

$x=\frac{1}{2}(a\pm\sqrt{a^2-4b^2})$, and $a-x=\frac{1}{2}(a\mp\sqrt{a^2-4b^2})$, are the parts required, and the two parts are the same, whether the upper or lower sign of the radical quantity be used. Thus, if the number a is 20, and b 8, the parts are 16 and 4, or 4 and 16.

The *forms* of these results enable us to determine the limits under which the problem is *possible;* for it is evident that if $4b^2$ be greater than a^2, $\sqrt{a^2-4b^2}$ becomes *imaginary.*

The extreme possible case will be, when $\sqrt{a^2-4b^2}=0$, in which case $x=\frac{1}{2}a$, and $a-x=\frac{1}{2}a$; also, $b^2=\frac{1}{4}a^2$, and $b=\frac{1}{2}a$.

In the following examples, that value of the unknown quantity only is given, which satisfies the conditions of the question in an arithmetical sense:

5. What two numbers are those whose sum is 20 and product 36? Ans. 2 and 18.

6. Divide 15 into two such parts that their product shall be to the sum of their squares as 2 to 5. Ans. 5 and 10.

7. Find a number such, that if you subtract it from 10, and multiply the remainder by the number itself, the product shall be 21. Ans. 7 or 3.

8. Divide 24 into two such parts that their product shall be equal to 35 times their difference. Ans. 10 and 14.

9. Divide the number 346 into two such parts that the sum of their square roots shall be 26. Ans. 11^2 and 15^2.

SUGGESTION.—Let $x=$ the square root of one of the parts, and $26-x$, of the other.

10. What number added to its square root gives 132?
Ans. 121.

11. What number exceeds its square root by $48\frac{3}{4}$?
Ans. $56\frac{1}{4}$.

12. What two numbers are those, whose sum is 41, and the sum of whose squares is 901? Ans. 15 and 26.

13. What two numbers are those, whose difference is 8, and the sum of whose squares is 544? Ans. 12 and 20.

14. A merchant sold a piece of cloth for \$24, and gained as much per cent. as the cloth cost him. Required the first cost. Ans. \$20.

15. Two persons, A and B, had a distance of 39 miles to travel, and they started at the same time; but A, by traveling $\frac{1}{4}$ of a mile an hour more than B, arrived one hour before him; find their rates of traveling.
Ans. A $3\frac{1}{4}$, B 3 mi. per hr.

16. A and B distribute \$1200 each among a number of persons; A gives to 40 persons more than B, and B gives \$5 apiece to each person more than A; find the number of persons. Ans. 120 and 80.

17. From two towns, distant from each other 320 miles, two persons, A and B, set out at the same instant to meet each other; A traveled 8 miles a day more than B, and the number of days before they met was equal to half the number of miles B went in a day; how many miles did each travel per day? Ans. A 24, B 16 mi.

18. A set out from C toward D, and traveled 7 miles a day. After he had gone 32 miles, B set out from D toward C, and went every day $\frac{1}{19}$ of the whole journey; and after he had traveled as many days as he went miles in one day, he met A. Required the distance from C to D.
Ans. 76, or 152 miles.

19. A grazier bought a certain number of oxen for \$240, and after losing 3 sold the remainder for \$8 a head more

than they cost him, thus gaining \$59 by his bargain. What number did he buy? Ans. 16.

20. Divide the number 100 into two such parts that their product may be equal to the difference of their squares. Ans. 38.197, and 61.803 nearly.

21. Two persons, A and B, jointly invested \$500 in business; A let his money remain 5 months, and B only 2, and each received back \$450, capital and profit. How much did each advance? Ans. A \$200, B \$300.

22. It is required to divide each of the numbers 11 and 17 into two parts, so that the product of the first parts of each may be 45, and of the second parts 48.

Ans. 5, 6, and 9, 8.

Represent the four parts by x, $11-x$, $\frac{45}{x}$, and $17-\frac{45}{x}$, and put the product of the second and fourth equal to 48.

23. Divide each of the numbers 21 and 30 into two parts, so that the first part of 21 may be three times as great as the first part of 30; and that the sum of the squares of the remaining parts may be 585.

Ans. 18, 3, and 6, 24.

24. Divide each of the numbers 19 and 29 into two parts, so that the difference of the squares of the first parts of each may be 72, and the difference of the squares of the remaining parts 180. Ans. 7, 12, and 11, 18.

DISCUSSION OF THE GENERAL EQUATION IN QUADRATICS.

234. The discussion of the general equation in quadratics consists in investigating its *general properties*, and in *interpreting the results* derived from making particular suppositions on the different quantities which it contains.

Taking the general form, (Art. 231,) $x^2+2px=q$, and completing the square, we have

$$x^2+2px+p^2=q+p^2.$$

Now, $x^2+2px+p^2=(x+p)^2$. For the sake of simplicity,

Put, . . . $q+p^2=m^2$; that is, $\sqrt{q+p^2}=m$.
Then, ... $(x+p)^2=m^2$;
Transposing, . $(x+p)^2-m^2=0$.

But, since the left member is the difference of two squares, it may be resolved into two factors (Art. 93); this gives

$$(x+p+m)(x+p-m)=0.$$

Now, this equation can be satisfied in *two* ways, and in *only* two; that is, by making either of the factors equal to 0. If we make the second factor equal to zero, we have

$$x+p-m=0;$$

Or, by transposing, $x=-p+m=-p+\sqrt{q+p^2}$.

If we make the first factor equal to zero, we have

$$x+p+m=0;$$

Or, by transposing, $x=-p-m=-p-\sqrt{q+p^2}$. Hence, we have

Property 1st.—*Every quadratic equation has two roots, (or values of the unknown quantity,) and only two.*

From the equation $(x+p+m)(x+p-m)=0$, we derive

Property 2d.—*Every affected equation, reduced to the form* $x^2+2px=q$, *may be decomposed into two binomial factors, of which the first term in each is* x, *and the second, the two roots with the signs changed.*

Thus, the two roots of the equation, $x^2-7x+10=0$, are $x=2$ and $x=5$. Hence, $x^2-7x+10=(x-2)(x-5)$.

It is now evident that the *direct* method of resolving a quadratic trinomial into its factors, is *to place it equal to zero, and then find the roots of the resulting equation.*

In this manner let the trinomials, Art. 94, be solved.

By reversing the operation, we can readily form an equation whose roots shall have any given values. Thus,

Let it be required to form an equation whose roots shall be -3 and 4.

We must have . . . $x=-3$, or $x+3=0$,

And . . . . $x=4$, or $x-4=0$.

Hence, ... $(x+3)(x-4)=0$;

Or, ... $x^2-x-12=0$;

Or, ... $x^2-x=12$, which is an equation whose roots are -3 and $+4$.

1. Find an equation whose roots are 4 and 5.
Ans. $x^2-9x=-20$.

2. Whose roots are $-\frac{1}{2}$ and $+\frac{1}{3}$. Ans. $x^2+\frac{1}{6}x=\frac{1}{6}$.

3. Find an equation, without fractional coëfficients, whose roots are $\frac{2}{3}$ and $\frac{4}{5}$. Ans. $15x^2-22x=-8$.

4. Find an equation whose roots are $m+n$ and $m-n$.
Ans. $x^2-2mx=n^2-m^2$.

Resuming the equation $x^2+2px=q$, and denoting the two roots by x' and x'', we have

$$x' = -p+\sqrt{q+p^2},$$
$$x''=-p-\sqrt{q+p^2}.$$

Adding, $x'+x''=-2p$. But, $-2p$ is the coëfficient of x, taken with a *contrary* sign. Hence, we have

Property 3d.—*The sum of the two roots of a quadratic equation, reduced to the form* $x^2+2px=q$, *is equal to the coëfficient of the first power of* x, *taken with a contrary sign.*

If we take the product of the roots, we have

$$x'x''=p^2 - (q+p^2)=-q.$$

But $-q$ is the known term of the equation, taken with a *contrary* sign. Hence, we have

Property 4th.—*The product of the two roots of a quadratic equation, reduced to the form* $x^2+2px=q$, *is equal to the known term taken with a contrary sign.*

In the preceding demonstrations, we have regarded $2p$ and q as both positive; but the same conclusions will be obtained by taking them both negative, or one positive and the other negative.

235. We shall now consider the *essential* sign, and numerical magnitude, of the roots in each of the four forms.

It is evident that the value of $\sqrt{q+p^2}$ must be *greater* than p, since the square root of p^2 alone, is p.

But the value of $\sqrt{-q+p^2}$ must be *less* than p, since it is the square root of a quantity less than p^2.

With these principles, a careful consideration of the roots, or values of x in each of the four different forms, will render the following conclusions evident:

1st form, . $x^2+2px=q$.

$$x'=-p+\sqrt{q+p^2}, \text{ and } x''=-p-\sqrt{q+p^2}.$$

The first root is essentially positive, the second essentially negative; and the first is numerically less than the second.

2d form, . $x^2-2px=q$.

$$x'=p+\sqrt{q+p^2}, \text{ and } x''=p-\sqrt{q+p^2}.$$

The first root is essentially positive, the second essentially negative; and the first is numerically greater than the second.

3d form, . . $x^2+2px=-q$.

$$x'=-p+\sqrt{-q+p^2}, \text{ and } x''=-p-\sqrt{-q+p^2}.$$

Both roots are essentially negative, and the first is numerically less than the second.

4th form, $x^2-2px=-q$.

$$x'=p+\sqrt{-q+p^2}, \text{ and } x''=p-\sqrt{-q+p^2}.$$

Both roots are essentially positive, and the first is numerically greater than the second.

236. We shall now proceed to show *when* the roots become imaginary, and *why*.

In the third and fourth forms, the radical part is $\sqrt{-q+p^2}$. Now, when q is *greater* than p^2, this is essentially negative, and the extraction of the root is impossible, (Art. 193.) Hence,

When the known term is negative, and greater than the square of half the coëfficient of the first power of x, *the roots are imaginary.*

To show why the roots are imaginary, we must prove that

When a number is divided into two equal parts, their product is greater than that of any other two parts into which the number can be divided.

Or, as the same principle may be otherwise expressed,

The product of any two unequal numbers is less than the square of half their sum.

Let $2p$ represent any number, and let the parts into which it is supposed to be divided, be $p+z$ and $p-z$. The product of these parts is

$$(p+z)(p-z)=p^2-z^2.$$

Now, this product is evidently the greatest, when z^2 is the least; that is, when $z^2=0$, or $z=0$. But when z is 0, the parts are p and p, which proves the proposition.

Now, it has been shown, (Art. 234, Properties 3d and 4th,) that $2p$ is equal to the *sum* of the two roots, and that q is equal to their *product*. But, when q is greater than p^2, we have the product of two numbers *greater* than the square of half their sum, which, by the preceding principle, is *impossible*.

If, then, any problem furnishes an equation of the form $x^2 \pm 2px = -q$, in which q is *greater* than p^2, the conditions are *incompatible* with each other. The following is an example:

Let it be required to divide the number 8 into two parts, whose product shall be 18.

Let x and $8-x$ represent the parts.

Then, $x(8-x)=18$; or $x^2-8x=-18$;

Whence, $x=4+\sqrt{-2}$, or $4-\sqrt{-2}$.

These expressions for the values of x, show that the problem is *impossible*, which is obviously true. By the preceding theorem, the greatest product of the parts of 8 is 16.

237. Examination of particular cases.

1st. If, in the third and fourth forms, where q is negative, we suppose $q=p^2$, the radical, $\sqrt{-q+p^2}$, becomes 0, and $x=-p$ in one, and $+p$ in the other.

It is then said, *the two roots are equal.*

In fact, if we substitute p^2 for q, the equation in the 3d form becomes

$$x^2+2px+p^2=0.$$

Hence, $\qquad (x+p)^2$, or, $(x+p)(x+p)=0.$

The first member is the *product of two equal factors*, either of which, placed equal to zero, gives the same value for x. A like result is obtained by substituting p^2 for q in the fourth form.

2d. If, in the general equation, $x^2+2px=q$, we suppose $q=0$, the two values of x reduce to,

$$x=-p+p=0, \text{ and } x=-p-p=-2p.$$

In fact, the equation is then of the form

$$x^2+2px=0, \text{ or } x(x+2p)=0,$$

which can be satisfied only by making

$$x=0, \text{ or } x+2p=0; \text{ whence, } x=0, \text{ or } x=-2p.$$

3d. If, in the general equation, $x^2+2px=q$, we suppose $2p=0$, we have

$$x^2=q; \text{ whence, } x=\pm\sqrt{q}.$$

In this case, *the two values of* x *are equal and have contrary signs, real,* if q is *positive*, as in the first and second forms, and *imaginary*, if q is *negative*, as in the third and fourth forms.

Under this supposition the equation contains only two terms, and belongs to the class treated of in Art. 228.

4th. If $2p=0$, and $q=0$, the equations reduce to $x^2=0$, and give the two values of x, in all the forms, each equal to 0.

238. There remains a singular case to be examined, which is sometimes met with in the solution of problems producing quadratic equations.

To discuss it, take the equation $ax^2+bx=c$.

Solving this equation, the values of x are

$$x=\frac{-b+\sqrt{b^2+4ac}}{2a}, \quad x=\frac{-b-\sqrt{b^2+4ac}}{2a}.$$

If, now, we suppose $a=0$, these values become

$$x=\frac{-b+b}{0}=\frac{0}{0}, \; x=\frac{-b-b}{0}=\frac{-2b}{0}=\infty.$$

That is, one value of x is *indeterminate* and the other *infinite*. (Arts. 136, 137.)

But if we suppose $a=0$ in the given equation, we have

$$bx=c, \text{ and } x=\frac{c}{b}.$$

We now propose to show that the *indeterminate* value is the same as the one last found, and that the *infinite* value simply expresses an *impossibility*.

If we multiply both terms of the second member of the equation

$$x=\frac{-b+\sqrt{b^2+4ac}}{2a}, \text{ by } -b-\sqrt{b^2+4ac}, \text{ we have}$$

$$x=\frac{b^2-(b^2+4ac)}{2a(-b-\sqrt{b^2+4ac})}=\frac{-4ac}{2a(-b-\sqrt{b^2+4ac})};$$

Or, by dividing both terms by $2a$, and making $a=0$,

$$x=\frac{-2c}{-b-\sqrt{b^2+4ac}}=\frac{-2c}{-2b}=\frac{c}{b}.$$

Hence, we see, that the value of $x=\frac{0}{0}$, is really $\frac{c}{b}$, and arises from having made the factor, $2a$, equal to zero. (See Art. 136.)

By supposing $a=0$, the equation $ax^2+bx=c$, reduces to $bx=c$, an equation of the *first* degree, which can have but *one* root.

The supposition that it has *two*, gives one value *infinite*, which is equivalent to saying, the equation has but one *finite* root.

If we had at the same time $a=0$, $b=0$, $c=0$, the equation would be altogether indeterminate. This is the only case of indetermination occurring in quadratic equations.

239. We shall now apply the principles above stated, in the discussion of the following

Problem of the Lights.—It is required to find, in a line BC, which joins two lights, B and C, of different intensities, a point which is illuminated equally by each.

It is a principle in optics, that *the intensity of the same light at different distances, is inversely as the square of the distance.*

Let a be the distance BC between the two lights.

Let b be the intensity of the light B at the distance of 1 ft. from B.

Let c be the intensity of the light C at the distance of 1 ft. from C.

Let P be the point required.

Let BP $=x$; then, CP $=a-x$.

By the principle above stated, since the intensity of the light B at the distance of 1 foot, is b, at 2, 3, 4, . . . feet, it must be $\frac{b}{4}, \frac{b}{9}, \frac{b}{16}$ . . . ; hence, the intensity of B and of C, at the distance of x and of $a-x$ feet, must be $\frac{b}{x^2}$ and $\frac{c}{(a-x)^2}$.

But, by the conditions of the problem, these two intensities are equal. Hence, we have for the equation of the problem,

$$\frac{b}{x^2}=\frac{c}{(a-x)^2}, \text{ which reduces to } \frac{(a-x)^2}{x^2}=\frac{c}{b};$$

Whence, $\frac{a-x}{x}=\frac{+\sqrt{c}}{\sqrt{b}}$, or $\frac{-\sqrt{c}}{\sqrt{b}}$.

This gives the following results :

1st. $x=\frac{a\sqrt{b}}{\sqrt{b}+\sqrt{c}}$; whence, $a-x=\frac{a\sqrt{c}}{\sqrt{b}+\sqrt{c}}$.

2d. $x=\frac{a\sqrt{b}}{\sqrt{b}-\sqrt{c}}$; whence, $a-x=\frac{-a\sqrt{c}}{\sqrt{b}-\sqrt{c}}$.

We shall now proceed to discuss these values.

I. Let $b>c$.

The first value of x, $\frac{a\sqrt{b}}{\sqrt{b}+\sqrt{c}}$, is positive, and less than a, for $\frac{\sqrt{b}}{\sqrt{b}+\sqrt{c}}$ is a proper fraction. Hence, this value gives for the point illuminated equally, a point P situated *between* B and C. We perceive, also, that the point P is nearer to C than B; for, since $b>c$, we have $\sqrt{b}+\sqrt{b}>\sqrt{b}+\sqrt{c}$, or $2\sqrt{b}>\sqrt{b}+\sqrt{c}$, and $\therefore \frac{\sqrt{b}}{\sqrt{b}+\sqrt{c}}>\frac{1}{2}$, and, consequently, $\frac{a\sqrt{b}}{\sqrt{b}+\sqrt{c}}>\frac{a}{2}$.

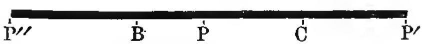
This is manifestly correct, for the required point must be nearer the light of less intensity. The corresponding value of $a-x$, $\frac{a\sqrt{c}}{\sqrt{b}+\sqrt{c}}$ is positive, and evidently less than $\frac{a}{2}$.

The second value of x, $\frac{a\sqrt{b}}{\sqrt{b}-\sqrt{c}}$, is positive, and greater than a; for $\sqrt{b}>\sqrt{b}-\sqrt{c}$; $\therefore \frac{\sqrt{b}}{\sqrt{b}-\sqrt{c}}>1$, and $\frac{a\sqrt{b}}{\sqrt{b}-\sqrt{c}}>a$.

This value gives a point P′, situated on the prolongation of BC, and in the *some direction* from B as before. In fact, since the two lights emit rays in all directions, there will be a point P′, to the right of C, and nearer the light of less intensity, which is illuminated equally by the two lights.

The second value of $a-x$, $\frac{-a\sqrt{c}}{\sqrt{b}-\sqrt{c}}$, is negative, as it ought to be, and represents the distance CP′, in the *opposite direction* from C, (Art. 47.)

II. Let $b<c$.



The first value of x, $\frac{a\sqrt{b}}{\sqrt{b}+\sqrt{c}}$, is positive, and less than $\frac{a}{2}$, for $\sqrt{b}+\sqrt{c}>\sqrt{b}+\sqrt{b}$; $\therefore \frac{\sqrt{b}}{\sqrt{b}+\sqrt{c}}<\frac{1}{2}$, and $\frac{a\sqrt{b}}{\sqrt{b}+\sqrt{c}}<\frac{a}{2}$.

The corresponding value of $a-x$, $\frac{a\sqrt{c}}{\sqrt{b}+\sqrt{c}}$, is greater than $\frac{a}{2}$, and positive. These values of x, and $a-x$, show that the point P is situated between B and C, but nearer to B than C. This is evidently a true result, since, under the present supposition, the intensity of the light B is less than that of the light C.

The second value of x, $\frac{a\sqrt{b}}{\sqrt{b}-\sqrt{c}}$, or $\frac{-a\sqrt{b}}{\sqrt{c}-\sqrt{b}}$, is essentially negative, and represents a point P″, in the *opposite direction* from B. As the intensity of the light B is now supposed to be less than that at C, there is, obviously, another such point of equal illumination.

The corresponding value of $a-x$, is $\frac{-a\sqrt{c}}{\sqrt{b}-\sqrt{c}}=\frac{a\sqrt{c}}{\sqrt{c}-\sqrt{b}}$. It is positive and greater than a, for $\sqrt{c}>\sqrt{c}-\sqrt{b}$ $\therefore$ $\frac{\sqrt{c}}{\sqrt{c}-\sqrt{b}}>1$, and $\frac{a\sqrt{c}}{\sqrt{c}-\sqrt{b}}>a$. This represents CP″, and is the *sum* of the distances CB and BP″, in the *same direction* from C as before.

III. Let $b=c$.

The first values of x and of $a-x$, reduce to $\frac{a}{2}$, which shows that the point illuminated equally is at the middle of the line BC, a result manifestly true, upon the supposition that the intensities of the two lights are *equal*.

The other two values are reduced to $\frac{a\sqrt{b}}{0}=\infty$. (Art. 136.) This result is manifestly true, for the intensities of the two lights being supposed equal, there is no point at any *finite* distance, except the point P, which is equally illuminated by both.

IV. Let $b=c$, and $a=0$.

The first system of values of x and $a-x$, become 0. This is evidently correct, for when the distance BC becomes 0, the distances BP and CP also become 0.

The second system of values of x and $a-x$, become $\frac{0}{0}$; this is the symbol of indetermination, (Art. 137.)

This result is also correct, for if the two lights are *equal*, and placed at the same point, *every point* on either side of them will be illuminated equally by each.

V. Let $a=0$, b not being $=c$.

All the values of x and $a-x$ reduce to 0; hence, there is *no point* equally illuminated by each. In other words, the solution of the problem fails in this case, as it evidently should.

This might also have been inferred from the original equation; for if we put $a=0$, $\frac{b}{x^2}=\frac{c}{(x-a)^2}$ becomes $\frac{b}{x^2}=\frac{c}{x^2}$, which can never be true except when $b=c$, as in Case IV.

239ᵃ. *Examples for discussion and illustration.*

1. Required a number such, that twice its square, increased by 8 times the number itself, shall be 90.

Ans. 5, or -9.

How may the question be changed, that the negative answer, taken positively, shall be correct in an arithmetical sense?

2. The difference of two numbers is 4, and their product 21. Required the numbers.

Ans. $+3$, $+7$, or -3 and -7.

3. A man bought a watch, which he afterward sold for \$16. His loss per cent. on the first cost of the watch, was the same as the number of \$'s which he paid for it. What did he pay for the watch? Ans. \$20, or \$80.

4. Required a number such, that the square of the number increased by 6 times the number, and this sum, increased by 7, the result shall be 2. Ans. $x=-1$, or -5.

What do the values of x show? How may the question be changed to be possible in an arithmetical sense?

5. Divide the number 10 into two such parts, that the product shall be 24. Ans. 4 and 6, or 6 and 4.

Is there more than one solution? Why?

6. Divide the number 10 into two such parts that the product shall be 26. Ans. $5+\sqrt{-1}$, and $5-\sqrt{-1}$.

What do these results show?

7. The mass of the earth is 80 times that of the moon, and their mean distance asunder 240000 miles. The attraction of gravitation being directly as the quantity of matter, and inversely as the square of the distance from the center of attraction, it is required to find at what point on the line passing through the centers of these bodies, the forces of attraction are equal.

Ans.	215865.5+	miles	from	the	earth,
and	24134.5—	"	"	"	moon.
Or,	270210.4+	"	"	"	earth,
and	30210.4+	"	beyond the moon from the earth.		

This question involves the same principles as the Problem of the Lights, and may be discussed in a similar manner. The required results, however, may be obtained directly from the values of x, page 208, calling $a=240000$, $b=80$, and $c=1$.

TRINOMIAL EQUATIONS.

240. A **Trinomial Equation** is one consisting of *three* terms, the general form of which is $ax^m+bx^n=c$.

Every trinomial equation of the form

$$x^{2n}+2px^n=q;$$

that is, every equation of three terms containing only *two* powers of the unknown quantity, and in which one of the exponents is *double* the other, can be solved in the same manner as an affected equation.

As an example, let it be required to find the value of x in the equation

$$x^4-2px^2=q.$$

Completing the square, $x^4-2px^2+p^2=q+p^2.$

$$x^2-p=\sqrt{q+p^2}.$$

$$x^2=+p\pm\sqrt{q+p^2}.$$

$$\therefore x=\pm\sqrt{p\pm\sqrt{q+p^2}}.$$

241. Binomial Surds.—Expressions of the form $A\pm\sqrt{B}$, like the value of x^2 just found, or of the form $\sqrt{A}\pm\sqrt{B}$, are called *Binomial Surds.*

The first of these forms, viz., $A\pm\sqrt{B}$, frequently results from the solution of trinomial equations of the *fourth* degree; and as it is sometimes possible to reduce it to a more simple form by extracting the square root, it is necessary to consider the subject here.

We shall first show that it is sometimes possible to extract the square root of $A\pm\sqrt{B}$, or to find the value of $\sqrt{A\pm\sqrt{B}}$.

Let us inquire how such binomial surds may arise from involution.

If we square $2\pm\sqrt{3}$, we have $4\pm4\sqrt{3}+3$, which, by reduction, becomes $7\pm4\sqrt{3}$. Hence, $\sqrt{7\pm4\sqrt{3}}=2\pm\sqrt{3}$. In the same way it may be shown that $\sqrt{5\pm2\sqrt{6}}=\sqrt{2}\pm\sqrt{3}$.

It thus appears that the form $A\pm\sqrt{B}$ may sometimes result from squaring a binomial of the form $a\pm\sqrt{b}$, or $\sqrt{a}\pm\sqrt{b}$, and uniting the extreme terms, which are necessarily rational, into one. In such cases, A is the sum of the squares of the two terms of the root, and $\sqrt{B}$ is twice their product.

To find the root, therefore, put $x^2+y^2=A$ and $2xy=\sqrt{B}$, and proceed to find x and y, the terms of the root. Thus,

Extract the square root of . . $7+4\sqrt{3}.$

Put . . . . . $x^2+y^2=7$ (1), and

$$2xy=4\sqrt{3}.$$

Adding, we have $x^2+2xy+y^2=7+4\sqrt{3}.$

Subtracting, we have $x^2-2xy+y^2=7-4\sqrt{3}.$

Extracting the root, $x+y=\sqrt{7+4\sqrt{3}}$ (2).

$x-y=\sqrt{7-4\sqrt{3}}$ (3).

Multiplying (2) by (3), $x^2-y^2=\sqrt{49-48}=\sqrt{1}=1$ (4).

By adding and subtracting (1) and (4), we have $2x^2=8$ .. $x=2$ and $2y^2=6$ .. $y=\sqrt{3}$. Hence, $2-\sqrt{3}$ is the root to be found.

1. Extract the square root of $15+6\sqrt{6}$. Ans. $3+\sqrt{6}$.
2. Of $34-24\sqrt{2}$. . . . . Ans. $4-3\sqrt{2}$.
3. Of $14\pm4\sqrt{6}$. . . . . . Ans. $\sqrt{2}\pm2\sqrt{3}$.

We shall now proceed to demonstrate more fully that the square root of $A\pm\sqrt{B}$ may always be found in a simple form, when A^2-B is a perfect square. To do this it is necessary to prove the following theorems:

Theorem I.—*The value of a quadratic surd can not be partly rational and partly irrational.*

For, if possible, let $\sqrt{x}=a-\sqrt{b}$; .. squaring both sides,

$x=a^2-2a\sqrt{b}+b$; .. $\sqrt{b}=\dfrac{x-a^2-b}{2a}$; that is, an irrational quantity is equal to a rational quantity, which is *impossible.*

Theorem II.—*In any equation of the form* $x\pm\sqrt{y}=a\pm\sqrt{b}$, *the rational quantities on opposite sides are equal, and also the irrational quantities.*

For if x does not $=a$, let $x=a+m$;

Therefore, $a+m+\sqrt{y}=a+\sqrt{b}$; $\therefore m+\sqrt{y}=\sqrt{b}$;
that is, the value of a quadratic surd is partly rational and partly irrational, which has been shown by Th. I, to be impossible; hence, $x=a$, and $\sqrt{y}=\sqrt{b}$.

We shall now proceed to find a formula for extracting the square root of $A+\sqrt{B}$.

Assume . . . . . $\sqrt{A+\sqrt{B}}=\sqrt{x}+\sqrt{y}$,

$$A+\sqrt{B}=x+y+2\sqrt{xy}, \text{ by squaring.}$$

By Th. II, $x+y=A$ (1); and $2\sqrt{xy}=\sqrt{B}$ (2);

Squaring equations (1) and (2), we have

$$\begin{aligned} x^2+2xy+y^2&=A^2 \\ 4xy\qquad&=B; \end{aligned}$$

Subtracting, $\quad x^2-2xy+y^2=A^2-B$; or, $(x-y)^2=A^2-B$.

Let A^2-B be a perfect square $=C^2$; then, $C=\sqrt{A^2-B}$.

Therefore, . . $(x-y)^2=C^2$, or $x-y=C$;

But, ... $x+y=A$;

Whence, . . . $x=\dfrac{A+C}{2}$; and $y=\dfrac{A-C}{2}$.

And . . $\sqrt{x}=\pm\sqrt{\dfrac{A+C}{2}}$; and $\sqrt{y}=\pm\sqrt{\dfrac{A-C}{2}}$.

Therefore, $\sqrt{x}+\sqrt{y}$, or $\sqrt{A+\sqrt{B}}=\pm\sqrt{\dfrac{A+C}{2}}\pm\sqrt{\dfrac{A-C}{2}}$.

Similarly, $\sqrt{x}-\sqrt{y}$, or $\sqrt{A-\sqrt{B}}=\pm\sqrt{\dfrac{A+C}{2}}\mp\sqrt{\dfrac{A-C}{2}}$.

Or, . $\quad \sqrt{A+\sqrt{B}}=\pm\left(\sqrt{\dfrac{A+C}{2}}+\sqrt{\dfrac{A-C}{2}}\right)$. (K.)

And $\quad \sqrt{A-\sqrt{B}}=\pm\left(\sqrt{\dfrac{A+C}{2}}-\sqrt{\dfrac{A-C}{2}}\right)$. (L.)

By substituting particular values for the general ones in these formulas, examples may be easily solved.

1. Extract the square root of $31+10\sqrt{6}$.

Here, $A=31$, $\sqrt{B}=10\sqrt{6}$; $\therefore A^2-B=C^2=961-600=361$; and $C=19$.

$\therefore A+C=50$, $A-C=12$.

Taking the formula and substituting, we have

$$\sqrt{A+\sqrt{B}}=\left(\sqrt{\frac{A+C}{2}}+\sqrt{\frac{A-C}{2}}\right).$$

$$\sqrt{31+10\sqrt{6}}=\sqrt{\frac{50}{2}}+\sqrt{\frac{12}{2}}=5+\sqrt{6}.$$

PROOF.—$(5+\sqrt{6})^2=25+10\sqrt{6}+6=31+10\sqrt{6}$.

2. Reduce $\sqrt{np+2m^2-2m\sqrt{np+m^2}}$, to its simplest form.

Here, $A=np+2m^2$, and $B=4m^2(np+m^2)$.

$A^2-B=n^2p^2$, and $C=np$, (formula L.)

$\therefore A+C=2np+2m^2$, $A-C=2m^2$ $\therefore$ $x=np+m^2$, $y=m^2$.

Formula (L) gives $\pm(\sqrt{np+m^2}-m)$.

3. Find the square root of $11+6\sqrt{2}$. Ans. $3+\sqrt{2}$.
4. Of $3\pm2\sqrt{2}$. . . . . Ans. $\sqrt{2}\pm1$.
5. Of $17+2\sqrt{60}$. . . . . . . Ans. $2\sqrt{3}+\sqrt{5}$.
6. Of $x-2\sqrt{x-1}$. . . . . . . Ans. $\sqrt{x-1}-1$.
7. Of $2a\sqrt{-1}$. (A=0.) . Ans. $\sqrt{a}(1+\sqrt{-1})$.
8. Of $x+y+z+2\sqrt{xz+yz}$. . Ans. $\sqrt{x+y}+\sqrt{z}$.

9. Find the value of $\sqrt{28+10\sqrt{3}}+\sqrt{67-16\sqrt{3}}$.

Ans. 13.

When A^2-B is not a perfect square, or when the binomial surd is of the form $\sqrt{A}\pm\sqrt{B}$, the root will be more complex than the original form.

REMARK.—By the above method the square root of any binomial or residual, as $a+b$, $a-b$, a^2+b^2, etc., may also be found, in a complex form.

242. We shall now resume the subject of *Trinomial Equations*. The general form of trinomial equations is $x^{2n}+2px^n=q$; but there are several varieties of this form,

of which the following are the principal: viz., $x+\sqrt{x}=q$, $x^4+px^2=q$, $x^n+px^{\frac{n}{2}}=q$, $x^{3n}+px^{\frac{3n}{2}}=q$, $x^{4n}+px^{2n}=q$, $(x^2+px+q)^2+b(x^2+px+q)=r$, and $(x^2+px+q)^{2n}+b(x^2+px+q)^n=k$.

Some of these varieties, if developed, would produce very complicated expressions, yet they may all be solved by the general method given in Art. 240.

1. Given $x^6-6x^3=16$, to find the value of x.

Assume, ... $x^3=y$; then, $x^6=y^2$, and

$y^2-6y \quad =16$;

Whence, ... $y=8$, or -2.

Therefore, . . . $x^3=8$, or -2; and $x=2$, or $-\sqrt[3]{2}$.

Or, the example may readily be solved without introducing a new letter. Thus,

Completing the square, $x^6-6x^3+9=25$.

Extracting the root, $x^3-3=\pm5$.

$x^3=8$, or -2, and $x=2$, or $-\sqrt[3]{2}$.

It will be shown hereafter, (Art. 396,) that in such examples as the preceding, there are four values of x not determined.

2. Given $5x-4\sqrt{x}=33$, to find the value of x.

Assume, ... $\sqrt{x}=y$; then, $x=y^2$, and

$5y^2-4y=33$;

Whence, ... $y=3$, or $-\frac{11}{5}$;

Consequently, . . $x=9$, or $\frac{121}{25}$.

3. Given $\sqrt{x+12}+\sqrt[4]{x+12}=6$, to find the value of x.

Assume, $\sqrt[4]{x+12}=y$; then, $\sqrt{x+12}=y^2$, and

$y^2+y=6$; whence, $y=2$, or -3;

Therefore, $\sqrt[4]{x+12}=2$, or -3.

Whence, $x+12=16$, or 81; and $x=4$, or 69.

Or, without introducing a new letter y, we may proceed to complete the square. Thus,

$$\sqrt{x+12}+\sqrt[4]{x+12}+\tfrac{1}{4}=6+\tfrac{1}{4}=\tfrac{25}{4};$$

Extracting the root, $\sqrt[4]{x+12}+\frac{1}{2}=\pm\frac{5}{2}$;

$$\sqrt[4]{x+12}=-\tfrac{1}{2}\pm\tfrac{5}{2}=+2, \text{ or } -3.$$

$$x+12=16, \text{ or } 81.$$

Whence, $\quad x=4$, or 69.

4. Given $3x^2+\sqrt{3x^2+1}=55$, to find the value of x.

Adding 1 to each member, the equation becomes

$$3x^2+1+\sqrt{3x^2+1}=56.$$

The equation may now be solved like the preceding.

The values of x are $+4$, -4, $+\sqrt{21}$, and $-\sqrt{21}$.

Find the values of x in each of the following examples:

5. $x^4-25x^2=-144$. . . . Ans. $x=\pm3$, or ±4.
6. $5x^4+7x^2=6732$. Ans. $x=\pm6$, or $\pm\frac{1}{10}\sqrt{-3740}$.
7. $9x^6-11x^3=488$. . . Ans. $x=2$, or $\frac{1}{3}\sqrt[3]{-183}$.
8. $x^3-x^{\frac{3}{2}}=15500$. . . Ans. $x=25$, or $(-124)^{\frac{2}{3}}$.
9. $x^{\frac{5}{6}}+x^{\frac{5}{3}}=1056$. . . Ans. $x=64$, or $(-33)^{\frac{6}{5}}$.
10. $x+5=\sqrt{x+5}+6$. . . . Ans. $x=4$, or -1.
11. $2\sqrt{x^2-3x+11}=x^2-3x+8$.

    Ans. $x=2$, 1, or $\frac{3}{2}\pm\frac{1}{2}\sqrt{-31}$.
12. $x^2-7x+\sqrt{x^2-7x+18}=24$.

    Ans. $x=9$, -2, or $\frac{1}{2}(7\pm\sqrt{173})$.
13. $(x^2-9)^2=3+11(x^2-2)$. Ans. $x=\pm5$, or ±2.
14. $\left(x+\dfrac{8}{x}\right)^2+x=42-\dfrac{8}{x}$.

    Ans. $x=4$, 2, or $\frac{1}{2}(-7\pm\sqrt{17})$
15. $x^4\left(1+\dfrac{1}{3x}\right)^2-(3x^2+x)=70$.

    Ans. $x=3$, $-3\frac{1}{3}$, or $\frac{1}{6}(-1\pm\sqrt{-251})$

16. $x\sqrt{\left(\frac{6}{x}-x\right)}=\frac{1+x^2}{\sqrt{x}}$. Ans. $x=\pm\sqrt{(1\pm\frac{1}{2}\sqrt{2})}$.

Sometimes it may be necessary to substitute a new letter *two* or more times, or to complete the square, without substitution, *three* or more times. The following is an example:

17. $x^4+5x^2+4\sqrt{x^4+5x^2}=60$.
Ans. $x=\pm2$, $\pm3\sqrt{-1}$, and $\pm\sqrt{\frac{5}{2}(-1\pm\sqrt{17})}$

243. Equations sometimes occur in which the *compound term* is not at first apparent, but which may be reduced to the form of a trinomial equation by the following method:

Extract the square root to two or three terms, and if we find a remainder (omitting known terms, if necessary,) which is any *multiple* or any *part* of the root already found, the given equation may be reduced to a trinomial, of which the *compound term* will be the root already found.

If the greatest exponent of the unknown quantity be not *even*, it must be made so by multiplying both members of the equation by the unknown quantity.

1. Given $x^3-4ax^2-2a^2x+12a^3=\frac{16a^4}{x}$, to find x.

Multiplying both sides by x, and transposing, we have

$$x^4-4ax^3-2a^2x^2+12a^3x-16a^4=0.$$

Proceeding to extract the square root, we have the following

OPERATION.

$$x^4-4ax^3-2a^2x^2+12a^3x-16a^4 \,|\, \underline{x^2-2ax.}$$

$$x^4$$

$$\underline{2x^2-2ax} \,|\, -4ax^3-2a^2x^2$$

$$\underline{-4ax^3+4a^2x^2}$$

Remainder, . $\quad -6a^2x^2+12a^3x-16a^4$;

Or, ... $\quad -6a^2(x^2-2ax)-16a^4$.

Hence, the given equation may be written thus:

$$(x^2-2ax)^2-6a^2(x^2-2ax)-16a^4=0.$$

Or, . . . $(x^2-2ax)^2-6a^2(x^2-2ax)=16a^4$.

Proceeding with the solution, we find

$$x=4a,\ -2a, \text{ or } a\pm a\sqrt{-1}.$$

2. $x^4-2x^3-2x^2+3x=108$.

Ans. $x=4$, -3, or $\frac{1}{2}(1\pm\sqrt{-35})$.

3. $x^4-2x^3+x=30$. Ans. $x=3$, -2, or $\frac{1}{2}(1\pm\sqrt{-19})$.

4. $x^3-6x^2+11x-6=0$. . . . Ans. $x=1$, 2, or 3.

5. $x^4-6x^3+5x^2+12x=60$.

Ans. $x=5$, -2, or $\frac{1}{2}(3\pm\sqrt{-15})$.

6. $x^4-8x^3+10x^2+24x=-5$.

Ans. $x=5$, -1, or $2\pm\sqrt{5}$.

7. $4x^4+\frac{x}{2}=4x^3+33$. Ans. $x=2$, $-\frac{3}{2}$, or $\frac{1}{4}(1\pm\sqrt{-43})$.

8. $\frac{x}{14}-\frac{30}{7x^3}+\frac{12+\frac{1}{2}x}{3x}=\frac{7}{2x^2}+1\frac{1}{6}$.

Ans. $x=4$, 3, or $\frac{1}{2}(7\pm\sqrt{69})$.

SIMULTANEOUS QUADRATIC EQUATIONS CONTAINING TWO OR MORE UNKNOWN QUANTITIES.

244. Quadratic Equations, containing two or more unknown quantities, may be divided into two classes, *pure* and *affected*.

Pure Equations embrace those that may be solved without completing the square.

Affected Equations embrace those in the solution of which it is necessary to complete the square.

The same equations may sometimes be solved by both methods.

PURE EQUATIONS.

245. Pure equations may in general be reduced to the solution of one of the following forms, or pairs of equations.

(1.) $\left.\begin{array}{r}x+y=a\\ xy=b\end{array}\right\}$. (2.) $\left.\begin{array}{r}x-y=c\\ xy=b\end{array}\right\}$. (3.) $\left.\begin{array}{r}x^2+y^2=a\\ x^2-y^2=b\end{array}\right\}$.

We shall explain the general method of solution in each of these cases.

To solve $x+y=a$ (1), and $xy=b$ (2), we must find $x-y$.

Squaring Eq. (1), . . $x^2+2xy+y^2=a^2$;
Multiplying Eq. (2) by 4, $\quad 4xy \quad =4b$;

Subtracting, . . $x^2-2xy+y^2=a^2-4b$,
Or, ... $(x-y)^2=a^2-4b$;
Whence, ... $x-y=\pm\sqrt{a^2-4b}$;
But, ... $x+y=a$;
Adding, and dividing by 2, $x=\frac{1}{2}a\pm\frac{1}{2}\sqrt{a^2-4b}$.
Subtracting, and dividing by 2, $y=\frac{1}{2}a\mp\frac{1}{2}\sqrt{a^2-4b}$.

The pair of equations (2) is solved in the same manner, except that in finding $x+y$, we must *add* 4 times the second equation to the square of the first.

The pair ot equations (3) is solved merely by adding and subtracting, then dividing by 2 and extracting the square root.

1. Given $x^2+y^2=25$, and $x+y=7$, to find x and y.

Squaring the 2d Eq., $x^2+2xy+y^2=49$;
But, ... $x^2 \quad +y^2=25$ (1).

Subtracting, . $2xy \quad =24$, (2).
Taking (2) from (1), $x^2-2xy+y^2=1$
Whence, ... $x-y=\pm1$ (3).
But, . . . $x+y=7$ (4).
Adding and subtracting (3) and (4), and dividing by 2,

$x=4$, or 3; and $y=3$, or 4.

2. Given $x^2+xy+y^2=91(1)$, and $x+\sqrt{xy}+y=13(2)$, to find x and y.

Divide Eq. (1) by (2), $x-\sqrt{xy}+y=7$. (3).
But, ... $x+\sqrt{xy}+y=13$. (2).

By subtracting, . . . $2\sqrt{xy}=6$.
Whence, ... $\sqrt{xy}=3$, and $xy=9$. (4).

By adding (2) and (3), . . . $x+y=10$. (5).

Squaring, (5), . $x^2+2xy+y^2=100$;

Multiplying (4) by 4, $4xy = 36$;

$$x^2-2xy+y^2=64, \therefore x-y=\pm 8.$$

But, $x+y=10$; whence, $x=9$, or 1; and $y=1$, or 9.

Equations of higher degrees than the second, that can be solved by simple methods, are usually classed with pure equations of the second degree.

3. Given $x^{\frac{1}{4}}+y^{\frac{1}{5}}=6$, and $x^{\frac{3}{4}}+y^{\frac{3}{5}}=126$, to find x and y.

In all cases of fractional exponents, the operations may be simplified by making such substitutions as will render the exponents *integral.* To do this, *put the lowest power of each unknown quantity equal to the first power of a new letter.*

In this example, let $x^{\frac{1}{4}}=P$, and $y^{\frac{1}{5}}=Q$; then, $x^{\frac{3}{4}}=P^3$, and $y^{\frac{3}{5}}=Q^3$. The given equations then become,

$$P+Q=6 \qquad (1),$$
$$P^3+Q^3=126 \qquad (2).$$

Dividing Eq. (2) by (1), $P^2-PQ+Q^2=21$;

Squaring Eq. (1), . . $P^2+2PQ+Q^2=36$;

Subtracting, . $3PQ=15$, $\therefore PQ=5$.

Having $P+Q=6$, and $PQ=5$, by the method explained in form (1), we readily find $P=5$, or 1; and $Q=1$, or 5.

Whence, $x=625$, or 1; and $y=1$, or 3125.

4. Given $(x-y)(x^2-y^2)=160$ (1),
$(x+y)(x^2+y^2)=580$ (2), to find x and y.

$x^3-x^2y-xy^2+y^3=160$ (1), by multiplying.

$x^3+x^2y+xy^2+y^3=580$ (2), " "

$2x^2y+2xy^2=420$ (3), by subtracting.

Add (3) to (2), $x^3+3x^2y+3xy^2+y^3=1000$.

Extract cube root, . $x+y=10$.

From (3), ... $xy(x+y)=210$; .. $xy=21$.

From $x+y=10$, and $xy=21$, we readily find $x=7$, or 3; and $y=3$, or 7.

Solve the following by the preceding or similar methods:

5. $\left.\begin{array}{l} x-y=2, \\ x^2+y^2=394. \end{array}\right\}$ . . . Ans. $x=15$, or -13; $y=13$, or -15.

6. $\left.\begin{array}{l} x^2+y^2=13, \\ xy=6. \end{array}\right\}$ . . . . Ans. $x=\pm 3$; $y=\pm 2$.

7. $\left.\begin{array}{l} 2x+y=7, \\ 4x^2+y^2=25. \end{array}\right\}$ . . . . . Ans. $x=2$, or $\frac{3}{2}$; $y=3$, or 4.

8. $\left.\begin{array}{l} x^2-y^2=16, \\ x-y=2. \end{array}\right\}$ . . . . Ans. $x=5$; $y=3$.

9. $\left.\begin{array}{l} x+y=11, \\ x^3+y^3=407. \end{array}\right\}$ . . . . . Ans. $x=7$, or 4; $y=4$, or 7.

10. $\left.\begin{array}{l} 7(x^3+y^3)=9(x^3-y^3), \\ x^2y-y^2x=16. \end{array}\right\}$ . . . . Ans. $x=4$; $y=2$.

11. $\left.\begin{array}{l} x^2+xy=84, \\ x^2-y^2=24. \end{array}\right\}$ . . . . . . Ans. $x=\pm 7$; $y=\pm 5$.

12. $\left.\begin{array}{l} x^3+y^3=152, \\ x^2-xy+y^2=19. \end{array}\right\}$ . . . . Ans. $x=5$, or 3; $y=3$, or 5.

13. $\left.\begin{array}{l} x^2+y^2+xy=208, \\ x+y=16. \end{array}\right\}$ . . . Ans. $x=12$, or 4; $y=4$, or 12.

14. $\left.\begin{array}{l} x^3-y^3=7xy, \\ x-y=2. \end{array}\right\}$ . . . . Ans. $x=4$, or -2; $y=2$, or -4.

15. $\left.\begin{array}{l} x^4+x^2y^2+y^4=91, \\ x^2+xy+y^2=13. \end{array}\right\}$ . . . Ans. $x=\pm 3$, or ± 1; $y=\pm 1$, or ± 3.

16. $\left.\begin{array}{l} x-y=\sqrt{x}+\sqrt{y}, \\ x^{\frac{3}{2}}-y^{\frac{3}{2}}=37. \end{array}\right\}$ . . . Ans. $x=16$, or 9; $y=9$, or 16.

17. $\left.\begin{array}{l} x^{\frac{1}{4}}+y^{\frac{1}{3}}=5, \\ x^{\frac{2}{4}}+y^{\frac{2}{3}}=13. \end{array}\right\}$ . . . . Ans. $x=16$, or 81; $y=27$, or 8.

18. $\left.\begin{array}{l} x^{\frac{1}{3}}+y^{\frac{1}{3}}=5, \\ x+y=35. \end{array}\right\}$ . . . . Ans. $x=8$, or 27; $y=27$, or 8.

19. $\left.\begin{array}{l}x^{\frac{1}{2}}+y^{\frac{1}{2}}=4,\\ x^{\frac{3}{2}}+y^{\frac{3}{2}}=28.\end{array}\right\}$ . . . . . . Ans. $x=9$, or 1; $y=1$, or 9.

20. $\left.\begin{array}{l}x^3+y^3=351,\\ xy=14.\end{array}\right\}$ . . . . . . Ans. $x=7$, or 2; $y=2$, or 7.

21. $\left.\begin{array}{l}x+y=4,\\ x^4+y^4=82.\end{array}\right\}$ . . . . . . Ans. $x=3$, or 1; $y=1$, or 3.

22. $\left.\begin{array}{l}x(y+z)=a,\\ y(x+z)=b,\\ z(x+y)=c.\end{array}\right\}$ Ans. $x=\pm\sqrt{\frac{(a+c-b)(a+b-c)}{2(b+c-a)}}$, $y=\pm\sqrt{\frac{(a+b-c)(b+c-a)}{2(a+c-b)}}$, $z=\pm\sqrt{\frac{(b+c-a)(a+c-b)}{2(a+b-c)}}$.

AFFECTED EQUATIONS.

246. The most general form of quadratic equations, containing two unknown quantities, is

$$ax^2+bxy+cx+dy^2+ey+f=0.$$

By arranging the terms according to the powers of x, and dividing by the coëfficient of the first term, two quadratic equations containing two unknown quantities, may be reduced to the following forms:

$$x^2+(a\,y+b\,)x+c\,y^2+d\,y+e\,=0 \qquad (1),$$
$$x^2+(a'y+b')x+c'y^2+d'y+e'=0 \qquad (2).$$

To find the values of either of the unknown quantities, we must eliminate the other. We shall now show that this operation produces an equation of the *fourth* degree.

By subtracting the second equation from the first, and making $a-a'=a''$, $b-b'=b''$, etc., we have

$$(a''y+b'')x+c''y^2+d''y+e''=0.$$

Whence, $x=$ . . . . $-\frac{c''y^2+d''y+e''}{a''y+b}$.

As this value of x contains y^2, that of x^2 will evidently contain y^4, which value of x^2, substituted in the first equation, necessarily gives rise to an equation of the *fourth* degree. Hence,

The solution of two quadratic equations, containing two unknown quantities, depends upon the solution of an equation of the fourth degree, containing one unknown quantity.

As there are no direct methods of solving equations of any higher degree than the second, those of the class now under consideration can not be solved except in particular cases, and then only by indirect methods, or special artifices.

We now proceed to point out some of these special cases, in addition to those already referred to in Arts. 242, 243, and 245, with some of the more common artifices employed.

247. There are two cases in quadratics which may *always* be solved as equations of the second degree, viz.:

Case I.—When one of the equations rises only to the *first* degree.

Given $\qquad ax+by=c \quad (1),$

$dx^2+exy+fy^2+gx+hy=k \quad (2)$, to find x and y.

From eq. (1), we may obtain a value of x in terms of y. Substituting this value, for x and x^2 in (2), the new equation will evidently contain only y and y^2.

Case II.—When both equations are *homogeneous*. (See Art. 30.)

Given $\quad a\,x^2+b\,xy+c\,y^2=d \quad (1),$

$a'x^2+b'xy+c'y^2=d' \quad (2)$, to find x and y.

Put $y=tx$, where t is a third unknown quantity, termed an *auxiliary* quantity. Substituting this value of y in the two equations, we have

$$a\,x^2+b\,tx^2+c\,t^2x^2=x^2(a\,+b\,t+c\,t^2)=d \qquad (3),$$
$$a'x^2+b'tx^2+c't^2x^2=x^2(a'+b't+c't^2)=d' \qquad (4).$$

From eq. (3), we find . . . $x^2=\dfrac{d}{a+bt+ct^2}$ (5).

From eq. (4), we find . . . $x^2=\dfrac{d'}{a'+b't+c't^2}$ (6).

Therefore, . . $\dfrac{d}{a+bt+ct^2}=\dfrac{d'}{a'+b't+c't^2}$.

Or, . . . $d(a'+b't+c't^2)=d'(a+bt+ct^2)$,

a quadratic equation, from which the value of t may be found, (Art. 231) and thence x from (5) or (6), and y from the equation $y=tx$.

248. When two quadratic equations are *symmetrical* with respect to the two unknown quantities; that is, when the two unknown quantities are *similarly* involved, they may frequently be solved by substituting for the unknown quantities the sum and difference of two others.

1. Given $x+y=a$ (1),
$x^5+y^5=b$ (2), to find x and y.

Let $x=s+z$, and $y=s-z$; then, $s=\dfrac{a}{2}$ (3),

$$x^5=s^5+5s^4z+10s^3z^2+10s^2z^3+5sz^4+z^5,$$
$$y^5=s^5-5s^4z+10s^3z^2-10s^2z^3+5sz^4-z^5;$$

$$x^5+y^5=2s^5+20s^3z^2+10sz^4=b.$$

By substituting the value of $s=\dfrac{a}{2}$, and reducing, we find

$$z^4+\frac{a^2}{2}z^2=\frac{16b-a^5}{80a}.$$

Completing the square, we find the value of z; and from (3), that of x and y.

249. An artifice that is often used with advantage, consists in adding such a number to both members of an equation as will render it a trinomial equation that can be resolved by completing the square, (Art. 240). The following is an example:

2. Given $\frac{x^2}{y^2}+\frac{y^2}{x^2}+\frac{x}{y}+\frac{y}{x}=\frac{27}{4}$ (1), and

$x^2+y^2=20$ (2), to find x and y.

Since $\left(\frac{x}{y}+\frac{y}{x}\right)^2=\frac{x^2}{y^2}+2+\frac{y^2}{x^2}$; add 2 to each side of eq. (1), and then $\frac{1}{4}$ to complete the square.

Therefore, $\left(\frac{x}{y}+\frac{y}{x}\right)^2+\left(\frac{x}{y}+\frac{y}{x}\right)+\frac{1}{4}=\frac{27}{4}+2+\frac{1}{4}=9$;

Whence, $\frac{x}{y}+\frac{y}{x}=\pm3-\frac{1}{2}=\frac{5}{2}$, or $-\frac{7}{2}$.

Let $\frac{x}{y}+\frac{y}{x}=\frac{5}{2}$; then, $\frac{x^2+y^2}{xy}$, or $\frac{20}{xy}=\frac{5}{2}$.

Whence, $xy=8$, and $2xy=16$.

From the equation $x^2+y^2=20$, and $2xy=16$, we readily find $x=\pm4$, and $y=\pm2$.

By taking $\frac{x}{y}+\frac{y}{x}=-\frac{7}{2}$, two other values of x and y may be found.

250. It is often of advantage to consider the sum, difference, product, or quotient of the two unknown quantities as a *single unknown quantity*, and find its value. Thus, in example 9, following, the value of xy should be found from the first equation, and in example 10, the value of $\frac{x}{y}$.

Other *auxiliaries* and expedients may frequently be employed with advantage, but their use can only be learned by experience, judgment, and tact.

NOTE.—In some of the examples all the values of the unknown quantities are not given; those omitted are generally imaginary.

3. $\left.\begin{array}{l} x^2+y^2+x+y=330, \\ x^2-y^2+x-y=150. \end{array}\right\}$ . . . Ans. $x=15$, or -16; $y=9$, or -10.

4. $\left.\begin{array}{l} x+4y=14, \\ y^2+4x=2y+11. \end{array}\right\}$ . . . . . Ans. $x=2$, or -46; $y=3$, or 15.

5. $\left.\begin{array}{l}2y-3x=14,\\3x^2+2(y-11)^2=14.\end{array}\right\}$ . . . Ans. $x=2$, or $1\frac{1}{5}$; $y=10$, or $8\frac{4}{5}$.

6. $\left.\begin{array}{l}x-y=2,\\\dfrac{x}{y}-\dfrac{y}{x}=1\frac{1}{15}.\end{array}\right\}$ . . . Ans. $x=5$, or $\frac{3}{4}$; $y=3$, or $-1\frac{1}{4}$.

7. $\left.\begin{array}{l}3x^2+xy=18,\\4y^2+3xy=54.\end{array}\right\}$ . . . Ans. $x=\pm2$, or $\pm2\sqrt{3}$; $y=\pm3$, or $\mp3\sqrt{3}$.

8. $\left.\begin{array}{l}x^2+xy=10,\\xy+2y^2=24.\end{array}\right\}$ . . . Ans. $x=\pm2$, or $\pm5\sqrt{2}$; $y=\pm3$, or $\mp4\sqrt{2}$.

9. $\left.\begin{array}{l}4xy=96-x^2y^2,\\x+y=6.\end{array}\right\}$ . . . Ans. $x=2$, 4, or $3\pm\sqrt{21}$; $y=4$, 2, or $3\mp\sqrt{21}$.

10. $\left.\begin{array}{l}\dfrac{x^2}{y^2}+\dfrac{4x}{y}=\dfrac{85}{9},\\x-y=2.\end{array}\right\}$ . . . Ans. $x=5$, or $\frac{17}{10}$; $y=3$, or $-\frac{3}{10}$.

11. $\left.\begin{array}{l}x^2y^2=180-8xy,\\x+3y=11.\end{array}\right\}$ . . . Ans. $x=5$, or 6; $y=2$, or $\frac{5}{3}$.

12. $\left.\begin{array}{l}x+y+\sqrt{x+y}=12,\\x^2+y^2=41.\end{array}\right\}$ . . . Ans. $x=5$, or 4; $y=4$, or 5.

13. $\left.\begin{array}{l}x+y+x^2+y^2=18,\\xy=6.\end{array}\right\}$ . . . Ans. $x=3$, 2, or $-3\pm\sqrt{3}$; $y=2$, 3, or $-3\mp\sqrt{3}$.

14. $\left.\begin{array}{l}x^2+3x+y=73-2xy,\\y^2+3y+x=44.\end{array}\right\}$ . . . Ans. $x=4$, or 16; $y=5$, or -7.

15. $\left.\begin{array}{l}xy+xy^2=12,\\x+xy^3=18.\end{array}\right\}$ . . . Ans. $x=2$, or 16; $y=2$, or $\frac{1}{2}$.

16. $\left.\begin{array}{l}x^2+y^2-x-y=78,\\x+y+xy=39.\end{array}\right\}$ . . . Ans. $x=9$, or 3; $y=3$, or 9.

17. $\left.\begin{array}{l}\left(\dfrac{3x}{x+y}\right)^{\frac{1}{2}}+\left(\dfrac{x+y}{3x}\right)^{\frac{1}{2}}=2,\\xy-(x+y)=54.\end{array}\right\}$ Ans. $x=6$, or $-4\frac{1}{2}$; $y=12$, or -9.

18. $\left.\begin{array}{l}\dfrac{y}{(x+y)^{\frac{3}{2}}}+\dfrac{\sqrt{x+y}}{y}=\dfrac{17}{4\sqrt{x+y}},\\x=y^2+2.\end{array}\right\}$ Ans. $x=6$, or 3; $y=2$, or 1.

QUESTIONS PRODUCING SIMULTANEOUS QUADRATIC EQUATIONS INVOLVING TWO OR MORE UNKNOWN QUANTITIES.

251.—1. There are two numbers, whose sum multiplied by the less is equal to 4 times the greater, but whose sum multiplied by the greater is equal to 9 times the less. What are the numbers? Ans. 3.6, and 2.4.

2. There is a number consisting of two digits, which being multiplied by the digit in the ten's place, the product is 46; but if the sum of the digits be multiplied by the same digit, the product is only 10. Required the number. Ans. 23.

3. What two numbers are those whose difference multiplied by the difference of their squares is 32, and whose sum multiplied by the sum of their squares is 272?
Ans. 5 and 3.

4. The product of two numbers is 10, and the sum of their cubes 133. Required the numbers. Ans. 2 and 5.

NOTE.—The preceding problems may be solved by pure equations.

5. What two numbers are those whose sum multiplied by the greater is 120, and whose difference multiplied by the less is 16? Ans. 2 and 10.

6. Find two numbers whose sum added to the sum of their squares is 42, and whose product is 15.
Ans. 3 and 5.

7. Find two numbers such, that their product added to their sum shall be 47, and their sum taken from the sum of their squares shall leave 62. Ans. 5 and 7.

8. Find two numbers such, that their sum, their product, and the difference of their squares, shall be all equal to each other. Ans. $\frac{3}{2}+\frac{1}{2}\sqrt{5}$, and $\frac{1}{2}+\frac{1}{2}\sqrt{5}$.

9. Find two numbers whose product is equal to the difference of their squares, and the sum of whose squares is equal to the difference of their cubes.

Ans. $\frac{1}{2}\sqrt{5}$, and $\frac{1}{4}(5+\sqrt{5})$.

10. A and B gained by trading \$100. Half of A's stock was less than B's by \$100, and A's gain was $\frac{3}{20}$ of B's stock. Supposing the gains in proportion to the stock, required the stock and gain of each.

Ans. A's stock \$600, B's \$400;
A's gain \$60, B's \$40.

11. The product of two numbers added to their sum is 23; and 5 times their sum taken from the sum of their squares leaves 8. Required the numbers. Ans. 2 and 7.

12. There are three numbers, the difference of whose differences is 5; their sum is 44, and continued product 1950; find the numbers. Ans. 25, 13, 6.

252. Formulæ.—A **General Solution** to a problem producing a quadratic equation, like one in simple equations, gives rise to a *formula*, (Art. 162,) which expressed in ordinary language, furnishes a *rule*. We shall illustrate the subject by a few examples.

Express each of the following formulæ in the form of a rule, and solve the numerical example by it:

1. Investigate a formula for finding two numbers, x and y, of which the sum of their squares is s, and difference of the squares d.

Ans. $x=\frac{1}{2}\sqrt{2(s+d)}$; $y=\frac{1}{2}\sqrt{2(s-d)}$.

Example.—Find two numbers such that the sum and difference of their squares are respectively 208 and 80.

Ans. 12 and 8.

2. Investigate a formula for finding two numbers, x and y, of which the difference is d, and the product p.

Ans. $x=\frac{1}{2}(d+\sqrt{d^2+4p})$; $y=\frac{1}{2}(-d+\sqrt{d^2+4p})$.

Ex.—A man is 8 years older than his wife, and the product of the numbers expressing the age of each is 2100. How old are they? Ans. Man 50, wife 42.

3. Investigate a formula for finding a number, x, of which the sum of the number and its square root is s.

Ans. $x=s+\frac{1}{2}-\sqrt{s+\frac{1}{4}}$.

Ex.—The sum of a number and its square root is 272; what is the number? Ans. 256.

4. The same when the difference of the number x, and its square root is d. Ans. $x=d+\frac{1}{2}+\sqrt{d+\frac{1}{4}}$.

Ex.—Find a number such that if its square root be subtracted from it, the remainder will be 132. Ans. 144.

5. Given $x+y=s$, and $xy=p$, to find the value of x^2+y^2, x^3+y^3, and x^4+y^4, in terms of s and p.

Ans. $x^2+y^2=s^2-2p$;

$x^3+y^3=s^3-3ps$;

$x^4+y^4=s^4-4ps^2+2p^2$.

Ex.—The sum of two numbers is 5, and their product 6. Required the sum of their squares, of their cubes, and of their fourth powers. Ans. 13, 35, and 97.

253. Special Solutions and Examples.—If an equation can be placed under the form

$$(x+a)\mathrm{X}=0,$$

in which X represents an expression involving x, at least one value of the unknown quantity may be found.

For since the equation will be satisfied by making either factor $=0$, we have $x+a=0$, and $\mathrm{X}=0$. Therefore, $x=-a$, is one solution of the equation, and the other values of x will be found by solving, if possible, the equation $\mathrm{X}=0$.

Thus, the equation $x^3-x^2-4x+4=0$, may be placed under the form $(x-2)(x^2+x-2)=0$. Hence, $x-2=0$, or $x=+2$; and, from the other factor, we find $x=+1$, or -2.

Skill in separating such an equation into its factors must be acquired by practice.

1. Given $x-1=2+\frac{2}{\sqrt{x}}$, to find x.

Since $x-1=(\sqrt{x}+1)(\sqrt{x}-1)$ and $2+\frac{2}{\sqrt{x}}=\frac{2}{\sqrt{x}}(\sqrt{x}+1)$;

Therefore, $(\sqrt{x}+1)(\sqrt{x}-1)=\frac{2}{\sqrt{x}}(\sqrt{x}+1)$;

Therefore, $\sqrt{x}+1=0$, and $x=(-1)^2=1$.

Also, $\sqrt{x}-1=\frac{2}{\sqrt{x}}$, by dividing by $\sqrt{x}+1$.

Whence, $\sqrt{x}=2$, or -1; and $x=4$, or 1.

2. $x^3-3x=2$. (Add $2x$ to each side.)

Ans. $x=-1$, or 2.

3. $x^2-\frac{2}{3x}=1\frac{4}{9}$. $\left(\text{Transpose } \frac{1}{9} \text{ and } \frac{2}{3x}.\right)$

Ans. $x=-\frac{2}{3}$, or $\frac{1}{3}(1\pm\sqrt{10})$.

4. $2x^3-x^2=1$. Ans. $x=1$, or $\frac{1}{4}(-1\pm\sqrt{-7})$.

5. $x^3-3x^2+x+2=0$. Ans. $x=2$, or $\frac{1}{2}(1\pm\sqrt{5})$.

6. $x^3=6x+9$. Ans. $x=3$, or $\frac{1}{2}(-3\pm\sqrt{-3})$.

7. $x+7x^{\frac{1}{3}}=22$. Ans. $x=8$, or $29\pm7\sqrt{-10}$.

$x+7x^{\frac{1}{3}}-22=(x-8)+7(x^{\frac{1}{3}}-2)$. $x^{\frac{1}{3}}-2$ is a divisor.

8. $x^4+\frac{13}{3}x^3-39x=81$.

Ans. $x=\pm3$, or $\frac{1}{6}(-13\pm\sqrt{-155})$.

An artifice that is frequently employed, consists in adding to each side of the equation, such a number or quantity as will render both sides perfect squares.

9. Given $x=\frac{12+8\sqrt{x}}{x-5}$, to find x.

Clearing of fractions, $x^2-5x=12+8\sqrt{x}$.

Add $x+4$ to each side, and extract the square root.

$$x-2=\pm(4+\sqrt{x}).$$

From which we easily find $x=9$, 4, or $\frac{1}{2}(-3\pm\sqrt{-7})$.

10. $x-3=\frac{3+4\sqrt{x}}{x}$.

Ans. $x=\frac{1}{2}(7\pm\sqrt{13})$, $\frac{1}{2}(-1\pm\sqrt{-3})$.

11. $\frac{49x^2}{4}+\frac{48}{x^2}-49=9+\frac{6}{x}$. Add $\frac{1}{x^2}$ to each side.

Ans. $x=2$, $-\frac{8}{7}$, or $\frac{1}{7}(-3\pm\sqrt{93})$.

12. $x^4+\frac{17x^3}{2}-34x=16$. Ans. $x=\pm2$, -8, or $-\frac{1}{2}$.

Transpose $34x$, and add $\left(\frac{17x}{4}\right)^2$ to each side.

13. $x^4\left(1+\frac{1}{3x}\right)^2-(3x^2+x)=70$.

Ans. $x=3$, $-3\frac{1}{3}$, or $\frac{1}{6}(-1\pm\sqrt{-251})$.

Divide by x^4, and add $\frac{9}{4x^4}$ to each side. (See Ex. 15, Art. 242.)

14. $\frac{18}{x^2}+\frac{81-x^2}{9x}=\frac{x^2-65}{72}$. Ans. 9, -4, or -9.

Multiply by 2, and add $\frac{8x}{36}+\frac{81}{36}$ to each side.

15. $27x^2-\frac{841}{3x^2}+\frac{17}{3}=\frac{232}{3x}-\frac{1}{3x^2}+5$.

Ans. $x=2$, $-\frac{14}{9}$, or $\frac{1}{9}(-2\pm\sqrt{-266})$.

Multiply both sides by 3, transpose $\frac{841}{x^2}$ and $\frac{1}{x^2}$, and add 1 to each side to complete the square.

We shall now present a few solutions giving examples of other artifices.

16. $\frac{1+x^4}{(1+x)^4}=a$; to find x.

$$1+x^4=a(1+x)^4=a(1+4x+6x^2+4x^3+x^4),$$
$$(1-a)(1+x^4)=4a(x+x^3)+6ax^2.$$

Dividing by x^2, $(1-a)\left(x^2+\frac{1}{x^2}\right)=4a\left(x+\frac{1}{x}\right)+6a$,

$$x^2+\frac{1}{x^2}-\frac{4a}{1-a}\left(x+\frac{1}{x}\right)=\frac{6a}{1-a},$$

$$\left(x+\frac{1}{x}\right)^2-\frac{4a}{1-a}\left(x+\frac{1}{x}\right)=\frac{6a}{1-a}+2=\frac{2+4a}{1-a}.$$

Complete the square, and find the value of $x+\frac{1}{x}$, which is $\frac{2a\pm\sqrt{2(1+a)}}{1-a}$; call this $2p$, and we then find $x=p\pm\sqrt{p^2-1}$.

17. $x^{x+y}=y^{4a}$, and $y^{x+y}=x^a$, to find x and y.

From 1st equation, $y=x^{\frac{x+y}{4a}}$.

From 2d equation, $y=x^{\frac{a}{x+y}}$;

Therefore, $x^{\frac{x+y}{4a}}=x^{\frac{a}{x+y}}$, and $\frac{x+y}{4a}=\frac{a}{x+y}$;

$(x+y)^2=4a^2$, or $x+y=2a$;

But, $x^a=y^{2a}$, since $x+y=2a$,

Therefore, $x=y^2$, and $y^2+y=2a$,

Whence, $y=\frac{1}{2}(-1\pm\sqrt{8a+1})$, and $x=\frac{1}{2}(4a+1\mp\sqrt{8a+1})$.

When two unknown quantities are found in an equation, in the form of $x+y$ and xy, it is generally expedient to put their sum $x+y=s$, and their product $xy=p$.

18. Given $(x+y)(xy+1)=18xy$ (1),
$(x^2+y^2)(x^2y^2+1)=208x^2y^2$ (2), to find x and y.

Let $x+y=s$, and $xy=p$; then,

$s(p+1)=18p$, (1), and
$(s^2-2p)(p^2+1)=208p^2$ (2).

From the square of (1), take (2), and after dividing by $2p$, we have

$s^2+p^2+1=58p$ (3).

But, . . $2s(p+1)=36p$, from (2),

And . . $2p=2p$.

Adding, . . . $(s+p+1)^2=96p$,

$s+p+1=4\sqrt{6p}$;

But, ... $p+1=\frac{18p}{s}$;

Therefore, . . . $s=4\sqrt{6p}-\frac{18p}{s}$, or $s^2-4s\sqrt{6p}=-18p$;

From which, . . $s=3\sqrt{6p}$, or $\sqrt{6p}$.

But, . . . $p+1=4\sqrt{6p}-s=3\sqrt{6p}$, or $\sqrt{6p}$.

Whence, . . . $p=26\pm\sqrt{675}$, or $2\pm\sqrt{3}$, and

$$s=\pm\sqrt{\{6(26\pm\sqrt{675})\}}, \text{ or } \pm\sqrt{\{6(2\pm\sqrt{3})\}}.$$

Having $x+y$ and xy, the values of x and y are easily found, (Art. 246); two of the values are $x=7\pm4\sqrt{3}$, $y=2\mp\sqrt{3}$.

19. $2(x+y)^3+1=(x^2+y^2)(xy+x^3+y^3)$ (1),
$x+y=3$ (2).
Ans. $x=2$, $y=1$.

20. $1+x^3=a(1+x)^3$. Ans. $x=\frac{1+2a\pm\sqrt{12a-3}}{2(1-a)}$.

21. $\frac{a}{x^2}-\frac{1}{x}\sqrt{x-2a-\frac{a}{x}}=1$.

Ans. $x=\frac{1}{4}\{1\pm\sqrt{1-8a}\pm\sqrt{2\pm2(1-8a)^{\frac{1}{2}}+8a}\}$.

22. $x+y+xy(x+y)+x^2y^2=85$, Ans. $x=6$, or 1.
$xy+(x+y)^2+xy(x+y)=97$. $y=1$, or 6.

23. $\frac{2c^2}{d^2}+\frac{ac}{d}-(a-b)(2c+ad)\frac{x}{d}=(a+b)\frac{cx}{d}-(a^2-b^2)x^2$.

Ans. $x=\frac{2c+ad}{(a+b)d}$, or $\frac{c}{(a-b)d}$.

24. $(x^3+1)(x^2+1)(x+1)=30x^3$.
Ans. $x=\frac{1}{2}(3\pm\sqrt{5})$.

25. $x^3+y^3=35$, Ans. $x=3$, 2, or $1\pm\frac{1}{2}\sqrt{22}$;
$x^2+y^2=13$. $y=2$, 3, or $1\mp\frac{1}{2}\sqrt{22}$.

26. $\dfrac{xyz}{x+y}=a$, Ans. $x=\sqrt{\left\{\dfrac{2abc(ac+bc-ab)}{(ab+ac-bc)(ab+bc-ac)}\right\}}$;

$\dfrac{xyz}{x+z}=b$, $\quad y=\sqrt{\left\{\dfrac{2abc(ab+bc-ac)}{(ac+bc-ab)(ab+ac-bc)}\right\}}$;

$\dfrac{xyz}{y+z}=c$. $\quad z=\sqrt{\left\{\dfrac{2abc(ab+ac-bc)}{(ac+bc-ab)(ab+bc-ac)}\right\}}$.

27. $(x^6+1)y=(y^2+1)x^3$,
$(y^6+1)x=9(x^2+1)y^3$.

Ans. $x=\frac{1}{2}\{\sqrt{\sqrt[3]{3}+3}+\sqrt{\sqrt[3]{3}-1}\}$;

$y=\frac{1}{2}\{\sqrt[3]{3}.\sqrt{\sqrt[3]{3}+3}\pm\sqrt{3\sqrt[3]{9}-1}\}$.

VIII. RATIO, PROPORTION, AND PROGRESSIONS.

254. Two quantities of the same kind may be compared in two ways. By considering,

1st. *How much* the one exceeds the other.

2d. *How many times* the one is *contained* in the other.

The first method is termed comparison by *difference;* the second, comparison by *quotient.* The first is sometimes called *Arithmetical ratio;* the second, *Geometrical ratio.*

If we compare 2 and 6, we find that 2 is *four less* than 6, or that 2 is *contained* in 6 *three* times. Also, the arithmetical ratio of a to b is $b-a$; the geometrical ratio of a to b is $\dfrac{b}{a}$.

The term Ratio, unless it is otherwise stated, always signifies geometrical ratio.

255. Ratio is the quotient which arises from dividing one quantity by another of the *same* kind. Thus, the ratio of 2 to 6 is 3, and the ratio of a to ma is m.

256. When two numbers, as 2 and 6, are compared, the *first* is called the *antecedent*, and the *second* the *consequent.* When spoken of as *one*, they are called a *couplet.* When spoken of as *two*, they are called the *terms* of the ratio.

Thus, 2 and 6 *together* form a couplet, of which 2 is the *first term*, and 6 the *second term.*

257. Ratio is expressed in two ways:

1st. In the form of a fraction, of which the *antecedent* is the *denominator*, and the *consequent* the *numerator.* Thus, the ratio of 2 to 6 is expressed by $\frac{6}{2}$; the ratio of a to b, by $\frac{b}{a}$.

2d. By placing two points between the terms. Thus, the ratio of 2 to 6, is written $2:6$; the ratio of a to b, $a:b$, etc.

258. The ratio of two quantities may be either a whole number, a common fraction, or an *interminate* decimal.

Thus, the ratio of 2 to 6 is $\frac{6}{2}$, or 3; of 10 to 4, is $\frac{2}{5}$.

The ratio of 2 to $\sqrt{5}$ is $\frac{\sqrt{5}}{2}$, or $\frac{2.236+}{2}$, or $1.118+$.

259. Since the ratio of two numbers is expressed by a fraction, of which the antecedent is the denominator, and the consequent the numerator, whatever is true with regard to a fraction is true with regard to the terms of a ratio. Hence,

1st. *To multiply the consequent, or divide the antecedent of a ratio by any number, multiplies the ratio by that number.*

2d. *To divide the consequent or to multiply the antecedent of a ratio by any number, divides the ratio by that number.*

3d. *To multiply or divide both the antecedent and consequent of a ratio by any number, does not alter the ratio.*

260. When the terms of a ratio are equal to each other, the ratio is said to be a ratio of *equality;* when the second term is greater than the first, a ratio of *greater inequality;* when it is less, a ratio of *less inequality.*

Thus, the ratio of 2 to 2 is a ratio of equality.

The ratio of 2 to 3 is a ratio of greater inequality.

The ratio of 3 to 2 is a ratio of less inequality.

Hence, a ratio of equality may be expressed by 1; a ratio of greater inequality, by a number greater than 1; and a ratio of less inequality, by a number less than 1.

261. When the corresponding terms of two or more ratios are multiplied together, the ratios are said to be *compounded*, and the result is termed a *compound ratio.*

Thus, the ratio of a to b, compounded with the ratio of c to d, is $\frac{b}{a}\times\frac{d}{c}=\frac{bd}{ac}$.

A ratio compounded of two equal ratios is called a *duplicate ratio;* one compounded of three equal ratios, a *triplicate ratio.*

Thus, the duplicate ratio of $\frac{b}{a}$ is $\frac{b}{a}\times\frac{b}{a}=\frac{b^2}{a^2}$; the triplicate ratio is $\frac{b^3}{a^3}$.

The ratio of the *square roots* of two quantities is called a *subduplicate ratio;* that of the *cube roots*, a *subtriplicate ratio.*

Thus, the subduplicate ratio of 4 to 9 is $\frac{3}{2}$; and that of a to b is $\frac{\sqrt{b}}{\sqrt{a}}$; the subtriplicate ratio of a to b is $\frac{\sqrt[3]{b}}{\sqrt[3]{a}}$.

262. Ratios may be compared with each other by reducing the fractions which represent them to a common denominator.

Thus, the ratio of 2 to 7 is greater than the ratio of 3 to 10, for the fractions $\frac{7}{2}$ and $\frac{10}{3}$, reduced to a common denominator, are $\frac{21}{6}$ and $\frac{20}{6}$, and the first is greater than the second.

PROPORTION.

263. Proportion is an equality of ratios; that is, *when two ratios are equal, their terms are said to be proportional.*

Thus, if the ratio of a to b is equal to the ratio of c to d; that is, if $\dfrac{b}{a}=\dfrac{d}{c}$; then, a, b, c, d, form a proportion.

Proportion is written in two ways:

1st. By placing a double colon between the ratios;

Thus, $a : b :: c : d$.

Read, a is to b as c is to d.

2d. By placing the sign of equality between the ratios;

Thus, $a : b = c : d$.

Read, the ratio of a to b equals the ratio of c to d.

From the preceding definition it follows, that when four quantities are in proportion, the second divided by the first, must give the same quotient as the fourth divided by the third. This is the primary *test* of the proportionality of four quantities.

Thus, if 3, 5, 6, 10, are the four terms of a proportion, so that $3 : 5 :: 6 : 10$, we must have $\frac{5}{3}=\frac{10}{6}$.

If these fractions are not equal to each other, the proportion is *false*.

Thus, the proportion $3 : 8 :: 2 : 5$ is *false*, since $\frac{8}{3}>\frac{5}{2}$.

REMARK.—The words *ratio* and *proportion* should not be confounded. Thus, two quantities are not in the *proportion* of 2 to 3, but in the *ratio* of 2 to 3. A ratio subsists between *two* quantities, a proportion between *four.*

264. Each of the four quantities in a proportion is called a *term.* The first and last terms are called the *extremes;* the second and third terms, the *means.*

265. Of four quantities in proportion, the first and third are called *the antecedents*, and the second and fourth, *the consequents* (Art. 257); and the last is said to be a fourth proportional to the other three taken in their order.

266. Three quantities are in proportion when the first has the same ratio to the second, that the second has to the third. The middle term is a *mean proportional* between the other two.

Thus, if . . . . $a : b : \ b : c$,

then b is a *mean proportional* between a and c; and c is called a *third proportional* to a and b.

When *several* quantities have the same ratio between each two that are consecutive, they are said to form a *continued proportion.*

267. Proposition I.—*In every proportion, the product of the means is equal to the product of the extremes.*

Let . . . . . . $a : b :: c : d$.

Since this is a true proportion, we must have (Art. 263)

$$\frac{b}{a}=\frac{d}{c}.$$

Clearing of fractions, $bc=ad$.

Illustration by numbers. $2 : 6 : 5 : 15$; and $6\times5=2\times15$.

Taking $bc=ad$, we find $d=\frac{bc}{a}$, $c=\frac{ad}{b}$, $b=\frac{ad}{c}$, $a=\frac{bc}{d}$. Hence,

If any three terms of a proportion be given, the remaining term may be found.

1. The first three terms of a proportion are $x+y$, x^2-y^2, and $x-y$; what is the fourth? Ans. $x^2-2xy+y^2$.

2 The 1st, 3d, and 4th terms of a proportion are $(m-n)^2$, m^2-n^2, and $m+n$; required the 2d. Ans. $m-n$.

3. The 1st, 2d, and 4th terms of a proportion are $\frac{a+\sqrt{b}}{a-\sqrt{b}}$, a^2-b, and $\frac{(a-\sqrt{b})(a^2-b)}{a+\sqrt{b}}$; required the 3d.

Ans. 1.

This proposition furnishes a more convenient *test* of proportionality than the method given in Art. 263.

Thus, $2:3::5:8$, is not a true proportion, since 3×5 is not equal to 2×8.

268. Proposition II.—Conversely, *If the product of two quantities is equal to the product of two others, two of them may be made the means, and the other two the extremes of a proportion.*

Let . . . . . . $bc=ad$.

Dividing each of these equals by ac, we have

$$\frac{bc}{ac}=\frac{ad}{ac},$$

Or, ... $\frac{b}{a}=\frac{d}{c}$.

That is (Art. 263), $a:b::c:d$.

By dividing each of the equals by ab, cd, bd, etc., we may have the proportion in other forms.

Or, since one member of the equation must form the extremes and the other the means, we have the following

Rule.—*Take either factor on either side of the equation for the first term of the proportion, the two on the other side for the second and third, and the remaining factor for the fourth.*

Thus, from each of the equations $bc=ad$, and $3\times12=4\times9$, we may have the *eight* following forms:

$a:b::c:d.$	$3:4::9:12.$
$a:c::b:d.$	$3:9::4:12.$
$d:b::c:a.$	$12:4::9:3.$
$d:c::b:a.$	$12:9::4:3.$
$b:a::d:c.$	$4:3::12:9.$
$b:d::a:c.$	$4:12::3:9.$
$c:a::d:b.$	$9:3::12:4.$
$c:d::a:b.$	$9:12::3:4.$

269. Proposition III.—*If three quantities are in proportion, the product of the extremes is equal to the square of the mean.*

If . . . . . . $a:b::b:c;$
Then, (Art. 267), . . . $ac=bb=b^2$.

It follows from Art. 268, that the converse of this proposition is also true.

Thus, if . . $ac=b^2$,
$a:b::b:c.$ Hence,

If the product of the first and third of three quantities is equal to the square of the second, the second is a mean proportional between the first and third.

270. Proposition IV.—*If four quantities are in proportion, they will be in proportion by* ALTERNATION; *that is, the first will be to the third as the second to the fourth.*

Let . . . . $a:b::c:d.$

Then, (Art. 263), . . $\frac{b}{a}=\frac{d}{c}.$

Multiply both sides by c, $\frac{bc}{a}=d;$

Divide both sides by b, $\frac{c}{a}=\frac{d}{b}.$

That is, (Art. 263), . $a:c::b:d.$

If . . . $2:6::10:30$; then, $2:10::6:30$.

271. Proposition V.—*If four quantities are in proportion, they will be in proportion by* INVERSION; *that is, the second will be to the first as the fourth to the third.*

Let . . . . $a : b :: c : d.$

Then, (Art. 263), ... $\frac{b}{a}=\frac{d}{c}$;

Inverting the fractions, . . $\frac{a}{b}=\frac{c}{d}$.

That is, (Art. 263), . . $b : a :: d : c.$

If . . . . $5 : 10 :: 6 : 12$; then, $10 : 5 :: 12 : 6$.

It follows from this proposition, that the equation $\frac{b}{a}=\frac{d}{c}$ may be converted into a proportion in either of two ways, thus:

$$a : b :: c : d, \text{ or } b : a :: d : c.$$

272. Proposition VI.—*If two sets of proportions have an antecedent and consequent in the one, equal to an antecedent and consequent in the other, the remaining terms will be proportional.*

Let . . . . $a : b :: c : d$ (1),

And . . . . . . $a : b :: e : f$ (2);

Then will . . . . $c : d :: e : f.$

From (1), $\frac{b}{a}=\frac{d}{c}$; from (2), $\frac{b}{a}=\frac{f}{e}$. Hence, $\frac{d}{c}=\frac{f}{e}$;

Which gives . . . $c : d :: e : f.$

If $4 : 8 :: 10 : 20$ and $4 : 8 :: 6 : 12$; then, $10 : 20 :: 6 : 12$.

273. Proposition VII.—*If four quantities are in proportion, they will be in proportion by* COMPOSITION; *that is, the sum of the first and second will be to the first or second, as the sum of the third and fourth is to the third or fourth.*

Let . . . . $a : b :: c : d$ (1),

Then will . . . $a+b : b :: c+d : d$.

From (1) . . . . . . $\frac{b}{a}=\frac{d}{c}$.

Inverting the fractions, . $\frac{a}{b}=\frac{c}{d}$.

Adding unity to both members, $\frac{a}{b}+1=\frac{c}{d}+1$.

Reducing to improper fractions, $\frac{a+b}{b}=\frac{c+d}{d}$.

Hence, (Art. 271), $a+b : b :: c+d : d$.

If $3 : 6 :: 9 : 18$; then, $3+6 : 6 :: 9+18 : 18$, or $9 : 6 :: 27 : 18$.

In a similar manner it may be shown that $a+b : a :: c+d : c$.

274. Proposition VIII.—*If four quantities are in proportion, they will be in proportion by* DIVISION; *that is, the difference of the first and second will be to the first or second, as the difference of the third and fourth is to the third or fourth.*

Let . . . . $a : b :: c : d$ (1),

Then will . . . . . . $a-b : b :: c-d : d$.

From (1), . . $\frac{b}{a}=\frac{d}{c}$.

Inverting the fractions, . $\frac{a}{b}=\frac{c}{d}$.

Subtracting unity from both members, $\frac{a}{b}-1=\frac{c}{d}-1$.

Reducing to improper fractions, . $\frac{a-b}{b}=\frac{c-d}{d}$.

This gives (Art. 271), $a-b : b :: c-d : d$.

If $18 : 6 :: 30 : 10$; then, $18-6 : 6 :: 30-10 : 10$, or $12 : 6 : 20 : 10$.

In a similar manner it may be shown that $a-b : a :: c-d : c$.

275. Proposition IX.—*If four quantities are in proportion, the sum of the first and second will be to their difference as the sum of the third and fourth is to their difference.*

Let . . . . $a : b :: c : d$ (1),
Then will . . . . $a+b : a-b :: c+d : c-d$.

From (1), by Composition and Division, (Arts. 273, 274,)

$$a+b : b :: c+d : d;$$

And . . . . $a-b : b :: c-d : d$.

By alternation, . $a+b : c+d :: b : d$;
And . . $a-b : c-d :: b : d$.

From which, (Art. 272), $a+b : c+d :: a-b : c-d$.
Or, by alternation, $a+b : a-b :: c+d : c-d$.
If $6 : 2 :: 12 : 3$; then, $6+2 : 6-2 :: 12+4 : 12-4$, or $8 : 4 :: 16 : 8$.

276. Proposition X.—*If four quantities are in proportion, like powers or roots of those quantities will also be in proportion.*

Let . . . . $a : b :: c : d$,
Then will . . . . $a^n : b^n :: c^n : d^n$.
From the 1st, ... $\frac{b}{a}=\frac{d}{c}$. Raising each of these equals
to the n^{th} power, ... $\frac{b^n}{a^n}=\frac{d^n}{c^n}$.
That is, ... $a^n : b^n :: c^n : d^n$,
Where n may be either a whole number or a fraction.
If $2 : 6 :: 10 : 30$; then, $2^2 : 6^2 :: 10^2 : 30^2$, or $4 : 36 :: 100 : 900$.
If $8 : 27 :: 64 : 216$; then, $\sqrt[3]{8} : \sqrt[3]{27} :: \sqrt[3]{64} : \sqrt[3]{216}$, or $2 : 3 :: 4 : 6$.

277. Proposition XI.—*If two sets of quantities are in proportion, the products of the corresponding terms will also be in proportion.*

Let . . . . $a : b :: c : d$ (1),
And . . . . $m : n :: r : s$ (2),
Then will . . . . . . $am : bn :: cr : ds$.

For from (1), $\frac{b}{a}=\frac{d}{c}$ (3); and from (2), $\frac{n}{m}=\frac{s}{r}$ (4).

Multiplying (3) by (4) $\frac{bn}{am}=\frac{ds}{cr}$; this gives, $am : bn :: cr : ds$.

If $3:9::2:6$, and $5:15::4:12$; then, $15:135::8:72$.

278. Proposition XII.—*In any number of proportions having the same ratio, any antecedent is to its consequent as the sum of all the antecedents is to the sum of all the consequents.*

Let . . . $a:b::c:d::m:n$, etc.

Then, ... $a:b::a+c+m:b+d+n$.

Since $a:b::c:d$, we have $bc=ad$ (Art. 267).

Since $a:b::m:n$, we have $bm=an$,

Also, ... $ab=ab$. The sum of these equalities gives . . . . $ab+bc+bm=ab+ad+an$.

Factoring, $b(a+c+m)=a(b+d+n)$.

This gives (Art. 268), $a:b::a+c+m:b+d+n$.

If $5:10::2:4:3:6$, etc.; then, $5:10::5+2+3:10+4+6$, or $5:10::10:20$.

EXERCISES IN RATIO AND PROPORTION.

1. Which is the greater ratio, that of 3 to 4, or 3^2 to 4^2? Ans. last.

2. Compound the duplicate ratio of 2 to 3; the triplicate ratio of 3 to 4; and the subduplicate ratio of 64 to 36. Ans. 1 to 4.

3. What quantity must be added to each of the terms of the ratio $m:n$, that it may become equal to $p:q$?

Ans. $\dfrac{mq-np}{p-q}$.

4. If the ratio of a to b is $2\frac{2}{3}$, what is the ratio of $2a$ to b, and of $3a$ to $4b$? Ans. $1\frac{1}{3}$, and $3\frac{5}{9}$.

5. If the ratio of a to b is $1\frac{2}{3}$, what is the ratio of $a+b$ to b, and of $b-a$ to a? Ans. $\frac{5}{8}$, and $\frac{3}{2}$.

6. If the ratio of m to n is $\frac{4}{7}$, what is the ratio of $m-n$ to $6m$, and also to $5n$? Ans. 14, and $6\frac{2}{3}$.

7. If the ratio of $5y-8x$ to $7x-5y$ is 6, what is the ratio of x to y? Ans. 7 to 11.

8. What eight proportions are deducible from the equation $ab=a^2-x^2$. Ans. $a : a+x : : a-x : b$,
$a : a-x : : a+x : b$,
$b : a+x : : a-x : a$, etc.

9. If $x^2+y^2=2ax$, what is the ratio of x to y?
Ans. $x : y : : y : 2a-x$.

10. Four given numbers are represented by a, b, c, d; what quantity added to each will make them proportionals?
Ans. $\dfrac{bc-ad}{a-b-c+d}$.

11. If four numbers are proportionals, show that there is no number which being added to each, will leave the resulting four numbers proportionals.

12. Find x in terms of y from the proportions $x : y : : a^3 : b^3$, and $a : b : : \sqrt[3]{c+x} : \sqrt[3]{d+y}$.

13. Prove that equal multiples of two quantities are to each other as the quantities themselves, or that $ma : mb : : a : b$.

14. Prove that like parts of two quantities are to each other as the quantities themselves, or that $\frac{a}{n} : \frac{b}{n} : : a : b$.

15. If $a : b : : c : d$, prove that $ma : mb : : nc : nd$, and also that $ma : nb : : mc : nd$, m and n being any multiples.

16. Prove that the quotients of the corresponding terms of two proportions are proportional.

279. The following examples are intended as exercises in application of the principles of proportion.

1. Resolve the number 24 into two factors, so that the sum of their cubes may be to the difference of their cubes as 35 to 19.

Let x and y denote the required factors; then, $xy=24$, and

$$x^3+y^3 : x^3-y^3 :: 35 : 19;$$

Therefore, (Art. 275), $2x^3 : 2y^3 :: 54 : 16;$

Or, . . . $x^3 : y^3 :: 27 : 8;$

Or, (Art. 276), . . $x : y :: 3 : 2.$

From which $y=\frac{2}{3}x$; then, substituting the value of y in the equation $xy=24$, we find $x=\pm 6$; hence, $y=\pm 4$.

2. Given $\dfrac{\sqrt[3]{x+1}+\sqrt[3]{x-1}}{\sqrt[3]{x+1}-\sqrt[3]{x-1}}=2$, to find x.

Resolving this equation into a proportion, we have

$$\sqrt[3]{x+1}-\sqrt[3]{x-1} : \sqrt[3]{x+1}+\sqrt[3]{x-1} :: 1 : 2;$$

$\therefore$ (Art. 275), $2\sqrt[3]{x+1} : 2\sqrt[3]{x-1} :: 3 : 1;$

Or, ... $\sqrt[3]{x+1} : \sqrt[3]{x-1} :: 3 : 1;$

Or, (Art. 276), . $x+1 : x-1 :: 27 : 1;$

(Art. 275), . $2x : 2 :: 28 : 26;$

Whence, . . . $52x=56$, or $x=1\frac{1}{13}$.

3. $x+y : x-y :: 3 : 1,$ $x^3-y^3=56.$ . . . . . . Ans. $x=4,$ $y=2.$

4. $x+y : x :: 7 : 5,$ $xy+y^2=126.$ . . . . . Ans. $x=\pm 15,$ $y=\pm 6.$

5. $(x+y)^2 : (x-y)^2 :: 64 : 1,$ $xy=63.$ . . Ans. $x=\pm 9,$ $y=\pm 7.$

6. $\dfrac{a-\sqrt{a^2-x^2}}{a+\sqrt{a^2-x^2}}=b.$ . . . . . . Ans. $x=\pm\dfrac{2a\sqrt{b}}{b+1}.$

7. $\dfrac{\sqrt{a+x}-\sqrt{a-x}}{\sqrt{a+x}+\sqrt{a-x}}=\dfrac{1}{b}.$ . . . . . Ans. $x=\dfrac{2ab}{b^2+1}.$

8. It is required to find two numbers whose product is 320, and the difference of whose cubes is to the cube of their difference, as 61 is to 1. Ans. 20 and 16.

280. Harmonical Proportion.—Three or four quantities are in *Harmonical Proportion* when the first has the same ratio to the last, that the difference between the first and second has to the difference between the last and the last except one.

Thus, a, b, c, are in harmonical proportion when $a : c :: a-b : b-c$; and a, b, c, d, when $a : d :: a-b : c-d$.

1. Let it be required to find a third harmonical proportional x, to two given numbers a and b.

We have, . $a : x :: a-b : b-x$;

Therefore, (Art. 267), $a(b-x)=x(a-b)$;

Whence, . . $x=\frac{ab}{2a-b}$.

2. Find a third harmonical proportional to 3 and 5.

Ans. 15.

3. Find a fourth harmonical proportional x, to three given numbers, a, b, and c.

Ans. $x=\frac{ac}{2a-b}$.

281. Variation, or, as it is sometimes termed, *General Proportion*, is merely an abridged form of common Proportion.

Variable Quantities are such as admit of various values in the same computation.

Constant, or Invariable Quantities have only one fixed value.

One quantity is said to *vary directly* as another, when the two quantities depend upon each other in such a manner, that if one be changed the other is changed *in the same ratio*.

Thus, the length of a shadow varies *directly* as the height of the object which casts it.

Such a relation between A and B is expressed thus,

$A \propto B$, the symbol $\propto$ being used instead of *varies*, or *varies as*.

282. There are four different kinds of Variation, which are distinguished as follows:

I. $A \propto B$. Here A is said to vary *directly* as B, or, simply A *varies as* B.

Ex.—If a man works for a certain sum per day, the amount of his wages *varies* as the number of days in which he works.

II. $A \propto \frac{1}{B}$. Here A is said to vary *inversely* as B.

Ex.—The *time* in which a man may perform a journey will vary *inversely* as the *rate* of traveling.

III. $A \propto BC$. Here A is said to vary as B and C *jointly*.

Ex.—The wages to be received by a workman will vary jointly as the *number* of days he works, and the *wages per day*.

IV $A \propto \frac{B}{C}$ Here A is said to vary *directly* as B, and *inversely* as C.

Ex.—The time occupied in a journey varies *directly* as the distance, and *inversely* as the rate of travel.

These four kinds of variation may be otherwise modified; thus, A may vary as the square or cube of B, inversely as the square or cube, directly as the square and inversely as the cube, etc.

Ex.—The intensity of the light shed by any luminous body upon an object will vary directly as the size of the luminous body, and inversely as the square of its distance from the object. (See Art. 238.)

In the following articles, A, B, C, represent corresponding values of any variable quantities, and a, b, c, any other corresponding values of the *same* quantities.

283. *If one quantity vary as a second, and that second as a third, the first varies as the third.*

Let $A \propto B$, and $B \propto C$, then shall $A \propto C$. For $A : a :: B : b$, and $B : b :: C : c$; therefore, (Art. 272), $A : a :: C : c$; that is, $A \propto C$.

In a similar manner it may be proved that if $A \propto B$, and $B \propto \frac{1}{C}$, that $A \propto \frac{1}{C}$.

284. *If each of two quantities vary as a third, their sum, or their difference, or the square root of their product, will vary as the third.*

Let $A \propto C$, and $B \propto C$; then, $A \pm B \propto C$; also, $\sqrt{AB} \propto C$.

By the supposition, . . .	$A : a :: C : c :: B : b$;
Therefore, ...	$A : a :: B : b$;
Alternately, (Art. 270), .	$A : B :: a : b$;
By Composition or Division,	$A \pm B : B :: a \pm b : b$;
Alternately, ...	$A \pm B : a \pm b :: B : b :: C : c$;
That is, ...	$A \pm B \propto C$.
Again, ...	$A : a :: C : c$;
And, ...	$B : b :: C : c$;
Therefore, (Art. 277), . .	$AB : ab :: C^2 : c^2$;
And, (Art. 276), ...	$\sqrt{AB} : \sqrt{ab} :: C : c$;
That is, ...	$\sqrt{AB} \propto C$.

By a similar method of reasoning, the following propositions may be proved:

285. *If one quantity vary as another, it will also vary as any multiple, or any part of the other.*

That is, if $A \propto B$; then, $A \propto mB$, or $\propto \frac{B}{m}$.

286. *If one quantity vary as another, any power or root of the former will vary as the same power or root of the latter.*

Let $A \propto B$; then, $A^n \propto B^n$, n being integral or fractional.

287. *If one quantity vary as another, and each of them be multiplied or divided by any quantity, variable or invariable, the products or quotients will vary as each other.*

Let $A \propto B$; then, $qA \propto qB$, and $\frac{A}{q} \propto \frac{B}{q}$.

288. *If one quantity vary as two others jointly, either of the latter varies as the first directly, and the other inversely.*

Let $A \propto BC$; then, $B \propto \frac{A}{C}$, and $C \propto \frac{A}{B}$.

289. *If* A *vary as* B, A *is equal to* B *multiplied by some constant quantity.*

Let $A \propto B$; then, $A = mB$.

If we know any corresponding values of A and B, the constant quantity m may be found.

290. In general, the simplest method of treating *variations*, is to convert them into *equations*.

1. Given that $y \propto$ the sum of two quantities, one of which varies as x, and the other as x^2, to find the corresponding equation.

Because one part $\propto x$, let this $= mx$,
and the other part $\propto x^2$, " " $= nx^2$.
Therefore, . . $y = mx + nx^2$,
where m and n are two unknown invariable quantities which can only be found when we know two pairs of corresponding values of x and y.

2. If $y=r+s$, where $r \propto x$ and $s \propto \frac{1}{x}$, and if, when $x=1$, $y=6$, and when $x=2$, $y=9$, what is the equation between x and y?

Let $r=mx$, and $s=\frac{n}{x}$ ∴ $y=mx+\frac{n}{x}$.

But if $x=1$, $y=6$, ∴ $6=m+n$;

And if $x=2$, $y=9$, ∴ $9=2m+\frac{n}{2}$.

Hence, $m=4$, $n=2$, and $y=4x+\frac{2}{x}$.

3. If $y \propto x$, and when $x=2$, $y=4a$; find the equation between x and y. Ans. $y=2ax$.

4. If $y \propto \frac{1}{x}$, and when $x=\frac{1}{2}$, $y=8$; find the equation between x and y. Ans. $y=\frac{4}{x}$.

5. If $y=$ the sum of two quantities, one of which varies as x, and the other varies inversely as x^2; and when $x=1$, $y=6$, and when $x=2$, $y=5$; find the equation between x and y. Ans. $y=2x+\frac{4}{x^2}$.

6. Given that $y=$ the sum of three quantities, of which the 1st is invariable, the 2d varies as x, and the 3d varies as x^2. Also, when $x=1, 2, 3$, $y=6, 11, 18$, respectively; find y in terms of x. Ans. $y=3+2x+x^2$.

7. Given that $s \propto t^2$, when f is constant; and $s \propto f$, when t is constant; also, $2s=f$, when $t=1$. Find the equation between f, s, and t. Ans. $s=\frac{1}{2}ft^2$.

Remarks.—1. The above examples may all be *proved*. Thus, if in Ex. 5, we put $x=1$ in the answer, y will equal 6. If we put $x=2$, $y=5$.

2. The Principles of Variation are extensively applied in mechanical philosophy.

ARITHMETICAL PROGRESSION.

291. An **Arithmetical Progression** is a series of quantities which increase or decrease by a *common difference.*

Thus, 1, 3, 5, 7, 9, etc., or 12, 9, 6, 3, etc., and a, $a+d$, $a+2d$, etc., a, $a-d$, $a-2d$, etc., are in Arithmetical Progression.

The series is said to be *increasing* or *decreasing*, according as d is positive or negative.

292. To investigate a rule for finding any term of an arithmetical progression, take the following series, in which the first line denotes the number of each term, the second an *increasing* arithmetical series, and the third a *decreasing* arithmetical series.

1	2	3	4	5	
a,	$a+d$,	$a+2d$,	$a+3d$,	$a+4d$,	etc.,
a,	$a-d$,	$a-2d$,	$a-3d$,	$a-4d$,	etc.

It is manifest that the coëfficient of d in any term is less by *unity* than the number of that term in the series; therefore, the n^{th} term $=a+(n-1)d$.

If we designate the n^{th} term by l, we have

$l=a+(n-1)d$, when the series is increasing, and

$l=a-(n-1)d$, when the series is decreasing. Hence,

Rule for finding Any Term of an Arithmetical Series.—*Multiply the common difference by the number of terms less one; when the series is increasing, add this product to the first term; when decreasing, subtract it from the first term.*

The equation $l=a+(n-1)d$, contains four variable quantities, any one of which may be found when the other three are known.

293. Having given the first term a, the common difference d, and the number of terms n, to find S, the sum of the series.

If we take any arithmetical series, as the following, and write the same series under it in an inverted order, we have

$$S= 1+3+5+7+9+11,$$
$$S=11+9+7+5+3+1.$$

Adding, . . $2S=12+12+12+12+12+12.$

$2S=12\times$ the number of terms, $=12\times 6=72.$

Whence, . . $S=\frac{1}{2}$ of $72=36$, the sum of the series.

To render this method general, let $l=$ the last term, and write the series both in a direct and inverted order.

Then, $S=a+(a+d)+(a+2d)+(a+3d). \quad . \quad +l,$

And, $S=l+(l-d)+(l-2d)+(l-3d). \quad . \quad . \quad +a.$

$2S=(l+a)+(l+a)+(l+a)+(l+a). \quad . \quad . \quad +(l+a),$

$2S=(l+a)$ taken as many times as there are terms (n) in the series.

Hence, ... $2S=(l+a)n;$

$$S=(l+a)\frac{n}{2}=\left(\frac{l+a}{2}\right)n. \quad \text{Hence,}$$

Rule for finding the Sum of an Arithmetical Series.—*Multiply half the sum of the two extremes by the number of terms.*

It also appears that

The sum of the extremes is equal to the sum of any other two terms equally distant from the extremes.

294. The equations $l=a+(n-1)d$, and $S=(a+l)\frac{n}{2}$, furnish the means of solving this general problem:

Knowing any three of the five quantities, a, d, l, n, S, which enter into an arithmetical series, to determine the other two.

The following table contains the results of the solution of all the different cases. As, however, it is not possible to retain these in

the memory, it is best, in ordinary cases, to solve all examples in Arithmetical Progression by the above two formulæ:

No.	Given.	Required.	Formulæ.
1.	a, d, n		$l=a+(n-1)d,$
2.	a, d, S		$l=-\frac{1}{2}d\pm\sqrt{\{2dS+(a-\frac{1}{2}d)^2\}},$
3.	a, n, S	l	$l=\frac{2S}{n}-a,$
4.	d, n, S		$l=\frac{S}{n}+\frac{(n-1)d}{2}.$
5.	a, d, n		$S=\frac{1}{2}n\{2a+(n-1)d\},$
6.	a, d, l		$S=\frac{l+a}{2}+\frac{l^2-a^2}{2d},$
7.	a, n, l	S	$S=(l+a)\frac{n}{2},$
8.	d, n, l		$S=\frac{1}{2}n\{2l-(n-1)d\}.$
9.	a, n, l		$d=\frac{l-a}{n-1},$
10.	a, n, S		$d=\frac{2(S-an)}{n(n-1)},$
11.	a, l, S	d	$d=\frac{l^2-a^2}{2S-l-a},$
12.	n, l, S		$d=\frac{2(nl-S)}{n(n-1)}.$
13.	a, d, l		$n=\frac{l-a}{d}+1,$
14.	a, d, S		$n=\frac{\pm\sqrt{(2a-d)^2+8dS}-2a+d}{2d},$
15.	a, l, S	n	$n=\frac{2S}{l+a},$
16.	d, l, S		$n=\frac{2l+d\pm\sqrt{(2l+d)^2-8dS}}{2d}$
17.	d, n, l		$a=l-(n-1)d,$
18.	d, n, S		$a=\frac{S}{n}-\frac{(n-1)d}{2},$
19.	d, l, S	a	$a=\frac{1}{2}d\pm\sqrt{(l+\frac{1}{2}d)^2-2dS},$
20.	n, l, S		$a=\frac{2S}{n}-l.$

1. Find the 15^{th} term of the series 3, 7, 11, etc.
Ans. 59.

Here, $a=3$, $n-1=14$, and $d=4$. Substituting these values in formula (1), we have $l=3+14\times4=3+56=59$.

2. Find the 20^{th} term of the series 5, 1, -3, etc.
Ans. -71.

3. Find the 8^{th} term of the series $\frac{2}{3}$, $\frac{7}{12}$, $\frac{1}{2}$, etc.
Ans. $\frac{1}{12}$

4. Find the 30^{th} term of the series -27, -20, -13, etc.
Ans. 176.

5. Find the n^{th} term of $1+3+5+7$. Ans. $2n-1$.
Of $2+2\frac{1}{3}+2\frac{2}{3}+$. . . . . Ans. $\frac{1}{3}(n+5)$.
Of $13+12\frac{2}{3}+12\frac{1}{3}+$. . . . Ans. $\frac{1}{3}(40-n)$.

6. Find the sum of $1+2+3+4$, etc., to 50 terms.

From formula (1), we find $l=50$. Substituting this in formula (2), we have $S=(1+50)25=1275$, Ans. Or, use formula 5.

7. Of $7+\frac{29}{4}+\frac{15}{2}+$, etc., to 16 terms. Ans. 142.

8. Of $12+8+4+$, etc., to 20 terms. Ans. -520.

9. Of $2+2\frac{1}{3}+2\frac{2}{3}+$, etc., to n terms. Ans. $\frac{n}{6}(n+11)$.

10. Of $\frac{1}{2}-\frac{2}{3}-\frac{11}{6}-$, etc., to n terms. A. $\frac{n}{12}(13-7n)$.

11. Of $\frac{n-1}{n}+\frac{n-2}{n}+\frac{n-3}{n}+$, etc., to n terms.
Ans. $\frac{n-1}{2}$.

12. If a falling body descends $16\frac{1}{12}$ feet the 1st sec., 3 times this distance the next, 5 times the next, and so on, how far will it fall the 30th sec., and how far altogether in half a min.? Ans. $948\frac{11}{12}$, and 14475 ft.

13. Two hundred stones being placed on the ground in a straight line, at the distance of 2 feet from each other;

how far will a person travel who shall bring them separately to a basket, which is placed 20 yards from the first stone, if he starts from the spot where the basket stands?

Ans. 19 miles, 4 fur., 640 ft.

14. Insert 3 arithmetical means between 2 and 14.

Here, $a=2$, $l=14$, and $n=5$. From formula (1), we obtain $d=3$. Hence, the three means will be 5, 8, and 11.

To solve this problem generally, let it be required to insert m arithmetical means between a and l.

Since there are m terms between a and l, we shall have $n=m+2$, and formula (1) becomes $l=a+(m+1)d$. Hence, $d=\frac{l-a}{m+1}$. Therefore,

The common difference will be equal to the difference of the extremes divided by the number of means plus one.

15. Insert 4 arithmetical means between 3 and 18.

Ans. 6, 9, 12, 15.

16. Insert 9 arithmetical means between 1 and -1.

Ans. $\frac{4}{5}$, $\frac{3}{5}$, etc., to $-\frac{4}{5}$.

17. How many terms of the series 19, 17, 15, etc., amount to 91? Ans. 13, or 7.

From (2) and (1), find n, or use formula 14. Explain this result.

18. How many terms of the series .034, .0344, .0348, etc., amount to 2.748? Ans. 60.

19. The sum of the first two terms of an arithmetical progression is 4, and the fifth term is 9; find the series.

Ans. 1, 3, 5, 7, 9, etc.

20. The first two terms of an arithmetical progression being together $=18$, and the next three terms $=12$, how many terms must be taken to make 28? Ans. 4, or 7.

21. In the series 1, 3, 5, etc., the sum of $2r$ terms : the sum of r terms : : x : 1; determine the value of x.

Ans. 4.

22. A sets out for a certain place, and travels 1 mile the first day, 2 the second, and so on. Five days afterward B sets out, and travels 12 miles a day. How long and how far must B travel to overtake A?

Ans. 3 days, or 10 days; and travel 36 miles, or 120 miles. Explain these results.

GEOMETRICAL PROGRESSION.

295. **A Geometrical Progression** is a series of terms, each of which is derived from the preceding, by multiplying it by a constant quantity, termed the *ratio.*

Thus, 1, 2, 4, 8, 16, etc., is an *increasing* geometrical progression, whose common ratio is 2.

Also, 54, 18, 6, 2, etc., is a *decreasing* geometrical progression, whose common ratio is $\frac{1}{3}$.

In general, a, ar, ar^2, ar^3, etc., is a geometrical progression, whose common ratio is r, and which is an *increasing* series when r is *greater* than 1; but a *decreasing* series when r is *less* than 1. It is evident that

In any given geometrical series, the common ratio will be found by dividing any term by the term next preceding.

296. To find the last term of a geometrical progression.

Let a denote the first term, r the common ratio, l the n^{th} term, and S the sum of n terms; then the respective terms of the series will be

$$\begin{array}{cccccccccc} 1, & 2, & 3, & 4, & 5, & \ldots & n-3, & n-2, & n-1, & n, \\ a, & ar, & ar^2, & ar^3, & ar^4 & \ldots & ar^{n-4}, & ar^{n-3}, & ar^{n-2}, & ar^{n-1}. \end{array}$$

That is, the exponent of r, in the *second* term, is 1, in the third term 2, in the *fourth* term 3, and so on. Hence, the n^{th} term of the series will be $l=ar^{n-1}$. Hence,

Rule for finding the Last Term of a Geometrical Series.—*Multiply the first term by the ratio raised to a power whose exponent is one less than the number of terms.*

Required to find the 6^{th} term of the geometrical progression whose first term is 7, and common ratio 2.

$$2^5=32; \text{ and } 7\times32=224, \text{ the } 6^{th} \text{ term.}$$

297. To find the sum of all the terms of a geometrical progression.

If we take the series, 1, 3, 9, 27, 81, and represent its sum by S; then, $S=1+3+9+27+81 \quad (a)$.

Multiplying by the ratio 3, $3S=3+9+27+81+243 \quad (b)$.

Subtracting (a) from (b), $3S-S=243-1$; whence, $S=121$.

To generalize this method, let a, ar, ar^2, ar^3, etc., be any geometrical series, and S its sum; then,

$$S=a+ar+ar^2+ar^3 \ldots +ar^{n-2}+ar^{n-1}.$$

Multiplying this equation by r, we have

$$rS=ar+ar^2+ar^3 \ldots +ar^{n-1}+ar^n.$$

Subtracting, $rS-S=ar^n-a$; whence, $S=\dfrac{a(r^n-1)}{r-1}$.

Since, ... $l=ar^{n-1}$, we have $rl=ar^n$;

Therefore, ... $S=\dfrac{ar^n-a}{r-1}=\dfrac{rl-a}{r-1}$. Hence,

Rule for finding the Sum of a Geometrical Series.—*Multiply the last term by the ratio, from the product subtract the first term, and divide the remainder by the ratio less one.*

Find the sum of 6 terms of the progression 3, 12, 48, etc.

$l=3\times4^5=3072$. ∴ $S=\dfrac{lr-a}{r-1}=\dfrac{3072\times4-3}{4-1}=4095$, Ans.

298. If the ratio r is less than 1, the progression is decreasing, and the last term l, or ar^{n-1}, is less than a. In

order that both terms of the fraction $\frac{rl-a}{r-1}$, or $\frac{ar^n-a}{r-1}$ may be positive, change the signs of the terms, (Art. 124), and $S=\frac{a-rl}{1-r}$, or $=\frac{a-ar^n}{1-r}$. Therefore, for finding the sum of the series, when the progression is decreasing,

Rule.—*Multiply the last term by the ratio, subtract the product from the first term, and divide the remainder by one minus the ratio.*

299. When the series is decreasing, and the number of terms infinite, l is *infinitely small*, or 0. Therefore, $rl=0$, and $S=\frac{a-rl}{1-r}$ becomes $S=\frac{a}{1-r}$. Hence,

Rule for finding the Sum of an Infinite Decreasing Series.—*Divide the first term by one minus the ratio.*

Find the sum of the infinite series $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+$, etc.

Here, $a=1$, $r=\frac{1}{2}$, and $S=\frac{a}{1-r}=\frac{1}{1-\frac{1}{2}}=2$, Ans.

That the sum of an *infinite* decreasing series may be *finite*, will easily appear from the following illustration:

Take a straight line, AK, and bisect it in B; bisect BK in C; CK in D, and so on continually; then will

AK=AB+BC+CD+, etc., *in infinitum*, =AB+$\frac{1}{2}$AB+$\frac{1}{4}$AB, etc., *in infinitum*, =2AB, which agrees with the example.

300. The equations, $l=ar^{n-1}$, and $S=\frac{ar^n-a}{r-1}$, furnish this general problem:

Knowing any three of the five quantities a, r, n, l, and S, of a geometrical progression, to determine the other two.

The following table contains all the values of each unknown quantity, or the equations from which it may be derived:

No.	Given.	Required.	Formulæ.
1.	a, r, n	l	$l=ar^{n-1}$,
2.	a, r, S		$l=\frac{a+(r-1)S}{r}$,
3.	a, n, S		$l(S-l)^{n-1}-a(S-a)^{n-1}=0$,
4.	r, n, S		$l=\frac{(r-1)Sr^{n-1}}{r^n-1}$
5.	a, r, n	S	$S=\frac{a(r^n-1)}{r-1}$,
6.	a, r, l		$S=\frac{rl-a}{r-1}$,
7.	a, n, l		$S=\frac{\sqrt[n-1]{l^n}-\sqrt[n-1]{a^n}}{\sqrt[n-1]{l}-\sqrt[n-1]{a}}$,
8.	r, n, l		$S=\frac{lr^n-l}{r^n-r^{n-1}}$.
9.	r, n, l	a	$a=\frac{l}{r^{n-1}}$,
10.	r, n, S		$a=\frac{(r-1)S}{r^n-1}$,
11.	r, l, S		$a=rl-(r-1)S$,
12.	n, l, S		$a(S-a)^{n-1}-l(S-l)^{n-1}=0$.
13.	a, n, l	r	$r=\sqrt[n-1]{\frac{l}{a}}$,
14.	a, n, S		$r^n-\frac{S}{a}r+\frac{S-a}{a}=0$,
15.	a, l, S		$r=\frac{S-a}{S-l}$,
16.	n, l, S		$r^n-\frac{S}{S-l}r^{n-1}+\frac{l}{S-l}=0$.
17.	a, r, l	n	$n=\frac{\log. l-\log. a}{\log. r}+1$,
18.	a, r, S		$n=\frac{\log. [a+(r-1)S]-\log. a}{\log. r}$,
19.	a, l, S		$n=\frac{\log. l-\log. a}{\log. (S-a)-\log. (S-l)}+1$,
20.	r, l, S		$n=\frac{\log. l-\log. [lr-(r-1)S]}{\log. r}+1$.

By observing, in any particular example, what are given and required, the proper formulæ may be selected from the above table. Nos. 3, 12, 14, and 16 may require the solution of an equation higher than the second degree. Nos. 17, 18, 19, and 20 are obtained by solving an exponential equation, (Art. 382) but are introduced here to render the table complete. The two formulæ

$$l=ar^{n-1} \text{ (1), and } S=\frac{lr-a}{r-1}, \text{ or, (Art. 298,) } \frac{a-lr}{1-r} \text{ (2),}$$

are, however, sufficient for the solution of all examples in Geometrical Progression, and may easily be retained in the memory.

1. Find the 8^{th} term of the series 5, 10, 20, etc.
Ans. 640.

2. The 7^{th} term of the series 54, 27, $13\frac{1}{2}$, etc.
Ans. $\frac{27}{32}$.

3. The 6^{th} term of the series $3\frac{3}{8}$, $2\frac{1}{4}$, $1\frac{1}{2}$, etc.
Ans. $\frac{4}{9}$.

4. The 7^{th} term of the series -21, 14, $-9\frac{1}{3}$, etc.
Ans. $-\frac{448}{243}$.

5. The n^{th} term of the series $\frac{1}{3}$, $\frac{1}{2}$, $\frac{3}{4}$, etc. Ans. $\frac{3^{n-2}}{2^{n-1}}$.

6. Find the sum of $1+3+9+$, etc., to 9 terms.

From (1), $l=1\times3^8=6561$. From (2), $S=\frac{3\times6561-1}{2}=9841$, Ans.

7. Of $1+4+16+$, etc., to 8 terms. Ans. 21845.

8. Of $8+20+50+$, etc., to 7 terms. Ans. $3249\frac{7}{8}$.

9. Of $1+3+9+$, etc., to n terms. Ans. $\frac{1}{2}(3^n-1)$.

10. Of $1-2+4-8+$, etc., to n terms. Ans. $\frac{1}{3}(1\mp2^n)$.

11. Of $x-y+\frac{y^2}{x}-\frac{y^3}{x^2}+$, etc., to n terms.
Ans. $\frac{x^2}{x+y}\left\{1-\left(-\frac{y}{x}\right)^n\right\}$.

12. The first term is 4, the last term 12500, and the number of terms 6. Required the ratio and the sum of all the terms. Ans. Ratio $=5$; sum $=15624$.

Find the sum of an infinite number of terms of each of the following series:

13. Of $\frac{2}{3}+\frac{1}{3}+\frac{1}{6}+$, etc. . . . . Ans. $\frac{4}{3}$.

14. Of $9+6+4+$, etc. . . . . Ans. 27.

15. Of $\frac{2}{3}-\frac{1}{3}+\frac{1}{6}-$, etc. . . . . Ans. $\frac{4}{9}$.

16. Of $a+b+\frac{b^2}{a}+\frac{b^3}{a^2}+$, etc. . . . . . Ans. $\frac{a^2}{a-b}$.

17. The sum of an infinite geometric series is 3, and the sum of its first two terms is $2\frac{2}{3}$; find the series.

Ans. $2+\frac{2}{3}+\frac{2}{9}+$ . . . or $4-\frac{4}{3}+\frac{4}{9}-$. . .

18. Find a geometric mean between 4 and 16. Ans. 8.

Here, $a=4$, $l=16$, and $n=3$; or, (Art. 269) the mean $=\sqrt{4\times16}$.

19. The first term of a geometric series is 3, the last term 96, and the number of terms 6; find the ratio, and the intermediate terms.

Ans. $r=2$. Int. terms, 6, 12, 24, 48.

If it be required to insert m geometrical means between two numbers, a and l, we have $n=m+2$; hence, $n-1=m+1$, and $r=\sqrt[m+1]{\frac{l}{a}}$. Or, we may employ formula (1).

20. Insert two geometric means between $\frac{16}{27}$ and 2.

Ans. $\frac{8}{9}$, $\frac{4}{3}$.

21. Insert 7 geometric means between 2 and 13122.

Ans. 6, 18, 54, 162, 486, 1458, 4374.

301. To find the value of *Circulating Decimals;* that is, decimals in which one or more figures are continually repeated.

In such decimals the ratio is $\frac{1}{10}$, $\frac{1}{100}$, $\frac{1}{1000}$, etc., according as one, two, or more figures recur. Thus,

$$.253131\ldots=\frac{25}{100}+\left(\frac{31}{10^4}+\frac{31}{10^6}+\frac{31}{10^8}+,\text{ etc.}\right)$$

The part within the parenthesis is an infinite series, having $a=\frac{31}{10000}$ and $r=\frac{1}{100}$. Hence, (Art. 299,) $S=\frac{31}{9900}$.

Therefore, $.253131\ldots=\frac{25}{100}+\frac{31}{9900}=\frac{2506}{9900}=\frac{1253}{4950}$.

This operation may be performed more simply, as follows:

Let	$S=.25313131\ldots$
Multiplying by 10000,	$10000S=2531.3131\ldots$
Dividing by 100, . .	$100S=\ \ 25.3131\ldots$
Subtracting, ...	$9900S=2506\ \therefore\ S=\frac{2506}{9900}$.

1. Find the value of $.6\dot{3}636\dot{3}$. . . . . Ans. $\frac{7}{11}$.
2. Find the value of $.54\dot{1}23123$. . . Ans. $\frac{18023}{33300}$.

302. Harmonical Progression.—Three or more quantities are said to be in Harmonical Progression, when their reciprocals are in arithmetical progression.

Thus, $1, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}$, etc.; and $\frac{1}{4}, \frac{2}{7}, \frac{1}{3}, \frac{2}{5}$, etc., are in harmonical progression, because their reciprocals

$1, 3, 5, 7$, etc.; and $4, 3\frac{1}{2}, 3, 2\frac{1}{2}$, etc.,

are in arithmetical progression.

303. Proposition.—*If three quantities are in harmonical progression, the first term is to the third as the difference of the first and second is to the difference of the second and third.*

For if a, b, c, are in harmonical progression, $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$, are in arithmetical progression; therefore,

$$\frac{1}{b}-\frac{1}{a}=\frac{1}{c}-\frac{1}{b}.\quad \text{Hence, multiplying by } abc,$$

$$ac-bc=ab-ac;\ \text{or}\ c(a-b)=a(b-c).$$

This gives (Art. 268), $a:c::a-b:b-c$; therefore,

A **Harmonical Progression** is a series of quantities in harmonical proportion (Art. 280); or such that if any three consecutive terms be taken, the first is to the third as the difference of the first and second is to the difference of the second and third.

Hence, all problems with respect to numbers in harmonical progression, may be solved by *inverting* them, and considering the reciprocals as quantities in arithmetical progression.

We give, however, below, two formulæ of frequent use:

1. Given the first two terms of a harmonical progression, a and b, to find the n^{th} term.

Here, a, b, and l, the first two and n^{th} terms become (Art. 302), $\frac{1}{a}$, $\frac{1}{b}$, and $\frac{1}{l}$ in formula (1) (Art. 294). Also, $d=\frac{1}{b}-\frac{1}{a}=\frac{a-b}{ab}$.

Therefore, $\frac{1}{l}=\frac{1}{a}+(n-1)\frac{a-b}{ab}=\frac{(n-1)a-(n-2)b}{ab}$;

Whence, $l=\frac{ab}{(n-1)a-(n-2)b}$.

By means of this formula, when any two successive terms of a harmonical progression are given, any other term may be found.

2. Insert m harmonic means between a and l.

Here, since $m=n-2$, and $m+1=n-1$, we have, as above,

$$\frac{1}{l}=\frac{1}{a}+(n-1)d, \text{ and } d=\frac{a-l}{(n-1)al}=\frac{a-l}{(m+1)al};$$

whence, the arithmetical progression is found; and by inverting its terms, the harmonicals are also found.

3. Insert two harmonic means between 3 and 12.

Ans. 4 and 6.

4. Insert two harmonic means between 2 and $\frac{1}{5}$.

Ans. $\frac{1}{2}$ and $\frac{2}{7}$.

5. The first term of a harmonic series is $\frac{1}{2}$, and the 6^{th} is $\frac{1}{12}$; find the intermediate terms.

Ans. $\frac{1}{4}$, $\frac{1}{6}$, $\frac{1}{8}$, $\frac{1}{10}$.

6. a, b, c, are in arithmetical progression, and b, c, d, are in harmonical progression; prove that $a : b :: c : d$.

PROBLEMS IN ARITHMETICAL AND GEOMETRICAL PROGRESSION.

304.—1. The sum of 5 numbers in arithmetical progression is 35, and the sum of their squares 335; find the numbers. Ans. 1, 4, 7, 10, 13.

Let $x-2y$, $x-y$, x, $x+y$, $x+2y$, be the numbers.

2. There are 4 numbers in arithmetic progression, and the sum of the squares of the extremes is 68, and of the means 52; find them. Ans. 2, 4, 6, 8.

Let $x-3y$, $x-y$, $x+y$, $x+3y$, be the numbers.

SUGGESTION.—When the number of terms in an arithmetic progression is *odd*, the common difference should be called y, and the middle term x; but when the number of terms is *even*, the common difference must be $2y$, and the two middle terms $x-y$ and $x+y$.

3. The sum of 3 numbers in arithmetical progression is 30, and the sum of their squares 308; find them.

Ans. 8, 10, 12.

4. There are 4 numbers in arithmetical progression, their sum is 26, and their product 880; find them.

Ans. 2, 5, 8, 11.

5. There are 3 numbers in geometrical progression, whose sum is 31; and the sum of the 1st and 2d . sum of 1st and 3d : : 3 : 13; find them. Ans. 1, 5, 25.

Let $x=$ 1st term and $y=$ ratio; then, xy and $xy^2=$ 2d and 3d terms.

6. The sum of the squares of 3 numbers in arithmetical progression is 83; and the square of the mean is greater by 4 than the product of the extremes; find them.

Ans. 3, 5, 7.

7. Find 4 numbers in arithmetical progression, such that the product of the extremes $=27$; of the means $=35$.

Ans. 3, 5, 7 9.

8. There are 3 numbers in arithmetical progression, whose sum is 18; but if you multiply the first term by 2, the second by 3, and the third by 6, the products will be in geometrical progression; find them.

Ans. 3, 6, 9.

9. The sum of the fourth powers of three successive natural numbers is 962; find them. Ans. 3, 4, 5.

10. The product of four successive natural numbers is 840; find them. Ans. 4, 5, 6, 7.

11. The product of four numbers in arithmetical progression is 280, and the sum of their squares 166; find them. Ans. 1, 4, 7, 10.

12. The sum of 9 numbers in arithmetical progression is 45, and the sum of their squares 285; find them.

Ans. 1, 2, 3, etc., to 9.

13. The sum of 7 numbers in arithmetical progression is 35, and the sum of their cubes 1295; find them.

Ans. 2, 3, etc., to 8.

14. Prove that when the arithmetical mean of two numbers is to the geometric mean : : 5 : 4; that one of them is 4 times the other.

15. The sum of 3 numbers in geometrical progression is 7; and the sum of their reciprocals is $\frac{7}{4}$; find them.

Ans. 1, 2, 4.

SUGGESTION.—In solving difficult problems in geometrical progression, it is sometimes preferable to express them by other forms. Thus, for 3 numbers, use x, $\sqrt{xy}$, y, or, x^2, xy, y^2; for four, $\frac{x^2}{y}$, x, y, $\frac{y^2}{x}$; for five, $\frac{x^3}{y}$, x^2, xy, y^2, $\frac{y^3}{x}$; for six, $\frac{x^3}{y^2}$, $\frac{x^2}{y}$, x, y, $\frac{y^2}{x}$, $\frac{y^3}{x^2}$.

In all these cases the product of the first and third of any three, taken consecutively, is equal to the square of the second. To find the ratio in each case, divide any expression by the preceding.

16. There are 4 numbers in geometrical progression, the sum of the first and third is 10, and the sum of the second and fourth is 30; find them. Ans. 1, 3, 9, 27.

17. There are 4 numbers in geometrical progression, the sum of the extremes is 35, the sum of the means is 30; find them. Ans. 8, 12, 18, 27.

18. There are 4 numbers in arithmetical progression, which being increased by 2, 4, 8, and 15 respectively, the sums are in geometrical progression; find them.

Ans. 6, 8, 10, 12.

19. There are 3 numbers in geometrical progression, whose continued product is 64, and the sum of their cubes 584; find them. Ans. 2, 4, 8.

IX. PERMUTATIONS, COMBINATIONS, AND BINOMIAL THEOREM.

305. The **Permutations** of quantities are the different orders in which they can be arranged.

Quantities may be arranged in sets of one and one, two and two, three and three, and so on.

Thus, if we have three quantities, a, b, c, we may arrange them in sets of *one*, of *two*, or of *three*, thus:

Of one,	a,	b,	c.
Of two,	ab, ac;	ba, bc;	ca, cb.
Of three,	abc, acb;	bac, bca;	cab, cba.

306. To find the number of permutations that can be formed out of n letters, taken *singly*, taken *two* together, *three* together. . . . and r together.

Let $a, b, c, d, \ldots k$, be the n letters; and let P_1 denote the whole number of permutations where the letters are taken *singly*; P_2 the whole number, taken 2 together . . . . and P_r the number taken r together.

The number of permutations of n letters taken singly, is evidently equal to the number of letters; that is,

$$P_1=n.$$

The number of permutations of n letters, taken two together, is $n(n-1)$. For since there are n quantities,

$$a,\ b,\ c,\ d,\quad .\quad k,$$

if we remove a, there will remain $(n-1)$ quantities. Writing a before each of these $(n-1)$ quantities, we shall have

$$ab,\ ac,\ ad\ \ .\ \ .\ \ .\ \ .\ \ ak.$$

That is, $(n-1)$ permutations in which a stands *first.*

In the same manner, there are $(n-1)$ permutations in which b stands first, and so of each of the remaining letters c, d, . k. Or, for n letters, there are $n(n-1)$ permutations taken *two* together.

That is, $P_2=n(n-1)$. Hence,

The number of permutations of n *letters taken two together, is equal to the number of letters, multiplied by the number less one.*

For example, if $n=4$, the number of permutations of a, b, c, d, taken two together, is $4\times3=12$. Thus, ab, ac, ad, || ba, bc, bd, || ca, cb, cd, || da, db, dc.

The number of permutations of n letters, taken *three* together, is $n(n-1)(n-2)$. For if we take $(n-1)$ letters,

b, c, d, . k, the number of permutations taken *two* together, by the last paragraph, is

$$(n-1)(n-2).$$

Let a be placed before each of these permutations; then, there are $(n-1)(n-2)$ permutations of n letters, taken *three* together, in which a stands first, and $(n-1)(n-2)$ permutations in which b stands *first;* and so for each of the n letters.

Hence, the whole number of permutations of n letters, taken *three* together, is $n(n-1)(n-2)$;

That is, $P_3=n(n-1)(n-2)$. Hence,

The number of permutations of n *letters taken three together, is equal to the number of letters, multiplied by the number less one, multiplied by the number less two.*

If $n=4$, the number of permutations of a, b, c, d, taken *three* together, is $4\times3\times2=24$. Thus,

abc, *abd*, *acb*, *acd*, *adb*, *adc*, *bac*, *bad*, *bca*, *bcd*, *bda*, *bdc*, *cab*, *cad*, *cba*, *cbd*, *cda*, *cdb*, *dab*, *dac*, *dba*, *dbc*, *dca*, *dcb*.

Following the same method, we prove that the number of permutations of n letters taken *four* together, is

$$P_4=n(n-1)(n-2)(n-3).$$

In each of the preceding results, the negative number in the last factor is *less by unity*, than the number of letters in each permutation.

Hence, for n things taken r together,

$$P_r=n(n-1)(n-2)\ .\ .\ .\ .\ .\ .\ .\ .\ (n-\overline{r-1})$$

306ª. Corollary.—If *all* the letters be taken together, then r becomes equal to n, and the last factor becomes 1;

That is, $P_n=n(n-1)(n-2)\ .\ .\ .\ .\ .\ 1.$

Or, inverting the order of the factors,

$P_n=1\times2\times3\ .\ .\ .\ .\ .\ .\ .\ (n-1)n.$ Hence,

The number of permutations of n *letters taken* n *together, is equal to the product of the natural numbers from* 1 *up to* n.

Ex.—The permutations of three letters, a, b, c, taken *three* together, is $1\times2\times3=6$.

307. If the *same* letter occur p times, the number of permutations in n letters, taken *all* together, is

$$\frac{1\times2\times3\ .\ .\ .\ .\ (n-1)n}{1\times2\times3\ .\ .\ .\ p}$$

Suppose these p letters to be all different. Then, for any particular position of the other letters, these p quantities, taken p together, will form $(1\times2\times3\ .\ .\ .\ .\ p)$ permutations from their interchange with each other; and when these letters are *alike*, these permutations are all reduced to *one*. And

As this is true for every position of the other letters, there will be altogether $(1\times2\times3\ .\ .\ .\ p)$ times fewer permutations when they are alike than when they are all different.

Thus, in the letters A, I, D, there are $1\times2\times3=6$ permutations taken *all* together, but if I becomes D, then three of these permutations become identical with the remaining three, and the whole number for ADD taken all together, is

$$\frac{1\times2\times3}{1\times2}=3.$$

307[a]. Corollary.—In like manner, if the same letter occur p times, another letter q times, a third letter r times, and so on, the number of permutations taken *all* together, is

$$\frac{1\times2\times3\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ (n-1)n}{(1\times2\ .\ .\ .\ p)(1\times2\ .\ .\ q)(1\times2\ .\ .\ r)\times,\ \text{etc.}}.$$

308. The **Combinations** of quantities are the *different* collections that can be formed out of them, without reference to the *order* in which they are placed.

Thus, ab, ac, bc, are the combinations of the letters a, b, c, taken *two* together; ab and ba, ac and ca, bc and cb, though different permutations, forming the same combination.

Proposition.—*To find the number of combinations that can be formed out of* n *letters, taken singly, taken two together, three together, ... and* r *together.*

Let C_1, C_2, . . . C_r denote the number of combinations of n things taken singly, taken two together, . . . and taken r together.

The number of combinations of n letters taken singly, is evidently n; that is,

$$C_1=n.$$

The number of *permutations* of n letters taken *two* together, is $n(n-1)$; but each combination, as ab, admits of (1×2) permuta-

tions, ab, ba; therefore, there are (1×2) times as many permutations as combinations. Hence,

$$C_2=\frac{n(n-1)}{1\times2}.$$

Again, in n letters taken *three* together, the number of permutations is $n(n-1)(n-2)$; but each combination of three letters, as abc, admits of $1\times2\times3$ permutations; therefore,

$$C_3=\frac{n(n-1)(n-2)}{1\times2\times3}.$$

So, for n letters, each of which contains r combinations,

$$C_r=\frac{n(n-1)(n-2)\ \ldots\ [n-(r-1)]}{1\times2\times3\ \ \ldots\ldots\ .r}$$

309. Intimately connected with the subject of the preceding articles, is that of the DOCTRINE OF CHANCES, or the CALCULUS OF PROBABILITIES. This, however, being too abstruse for an elementary treatise, is omitted in this work.

1. How many permutations of 2 letters each can be formed out of the letters a, b, c, d, e? How many of 3? Of 4? Ans. (1) 20. (2) 60. (3) 120.

2. How many combinations of 2 letters each can be formed out of the letters a, b, c, d, e? How many of 3? Of 4? Of 5? Ans. (1) 10. (2) 10. (3) 5. (4) 1.

3 In how many ways, taken all together, may the letters in the word NOT be written? In the word HOME? Ans 6, and 24.

4. How often can 6 persons change their places at dinner, so as not to sit twice in the same order? Ans. 720.

5. In how many different ways, taken all together, can the 7 prismatic colors be arranged? Ans. 5040.

6. In how many different ways can 6 letters be arranged when taken singly, 2 by 2, 3 by 3, and so on, till they are all taken? Ans. 1956.

SUGGESTION.—Take the sum of the different permutations.

7. How many different products can be formed with any two of the figures 3, 4, 5, 6? Ans. 6.

8. The number of permutations of n things taken 4 together $=$ 6 times the number taken 3 together; find n.
Ans. $n=9$.

9. How many different sums of money can be formed with a cent, a three cent piece, a half dime, and a dime?
Ans. 15.

SUGGESTION.—Take the sum of the different combinations of 4 things taken singly, 2 together, 3 together, and 4 together.

10. With the addition of a twenty-five cent piece, and a half dollar, to the coins in the last example, how many different sums of money may be formed? Ans. 63.

11. At an election, where every voter may vote for any number of candidates not greater than the number to be elected, there are 4 candidates and only 3 persons to be chosen; in how many ways may a man vote? Ans. 14.

12. On how many nights may a different guard of 4 men be posted out of 16? and on how many of these will any particular man be on guard? Ans. 1820, and 455.

13. How many changes may be rung with 5 bells out of 8, and how many with the whole peal?
Ans. 6720, and 40320.

14. Out of 17 consonants and 5 vowels, how many words can be formed, having two consonants and one vowel in each? Ans. 4080

BINOMIAL THEOREM,

WHEN THE EXPONENT IS A POSITIVE INTEGER.

310. We have already explained (Art. 172) the method of finding any power of a binomial, by repeated multiplication, and by Newton's Theorem, as proved experimentally.

We shall now proceed from the theory of Combinations (Art. 308), to demonstrate the *Binomial Theorem* in its most general form.

The Binomial Theorem teaches the method of developing into a series any binomial whose index is either integral or fractional, positive or negative; as,

$$(a+x)^n,\ (a+x)^{-n},\ (a+x)^{\frac{n}{m}},\ (a+x)^{-\frac{n}{m}},$$

where a or x may be either plus or minus.

The following investigation applies only to the case where the exponent is *positive* and *integral;* the other cases will be considered hereafter. (See Art. 319.)

By actual multiplication, it appears that

$$(x+a)(x+b)=x^2\begin{array}{r|l}+a & x+ab.\\ +b & \end{array}$$

In like manner, $(x+a)(x+b)(x+c)$

$$=x^3\begin{array}{r|r|l}+a & x^2+ab & x+abc.\\ +b & +ac & \\ +c & +bc & \end{array}$$

Also, $(x+a)(x+b)(x+c)(x+d)$

$$=x^4\begin{array}{r|r|r|l}+a & x^3+ab & x^2+abc & x+abcd.\\ +b & +ac & +abd & \\ +c & +ad & +acd & \\ +d & +bc & +bcd & \\ & +bd & & \\ & +cd & & \end{array}$$

An examination of either of these products shows that it is composed of a series of descending powers of x, and of certain coëfficients, formed according to the following law:

1st. The *exponent* of the highest power of x is found in the first term, and is the same as the *number* of binomial factors, and the other exponents of x decrease by 1 in each succeeding term.

2d. The *coëfficient* of the first term is 1; of the second, the sum of the quantities a, b, c, etc.; of the third, the

sum of the products of every two of the quantities a, b, c, etc.; of the fourth, the sum of the products of every three, and so on; and of the last, the product of all the n quantities a, b, c, etc.

Suppose, then, this law to hold for the product of n binomial factors $x+a$, $x+b$, $x+c$, . . . $x+k$; so that $(x+a)(x+b)(x+c)$ . . . $(x+k)=x^n+Ax^{n-1}+Bx^{n-2}+Cx^{n-3}+ \ldots +K$,

Where . . . $A=a+b+c+ \ldots +k$.

$B=ab+ac+ad+ \ldots$

$C=abc+abd+ \ldots$

Etc. = etc . . . .

$K=abcd \ldots k$.

If we multiply both sides of this equation by a new factor $x+l$, we have

$$\begin{array}{l}(x+a)(x+b)(x+c) \ldots (x+k)(x+l) \\ =x^{n+1}\begin{array}{l|l|l|l}+A & x^n+B & x^{n-1}+C & x^{n-2} \ldots \\ +l & +Al & +Bl & \ldots +Kl.\end{array}\end{array}$$

Here, . $A+l=a+b+c+ \ldots +k+l$;

$B+Al=ab+ac+ad \ldots +al+bl \ldots +kl$.

Etc. = etc. .

$Kl=abcd. \ldots kl$.

It is evident that the same law, as above stated, still holds.

Hence, if the law holds when n binomial factors are multiplied together, it will hold when $n+1$ factors are taken; but it has been shown, by actual multiplication, to hold up to 4 factors; therefore, it is true for $4+1$, that is, 5; and if for 5, then for $5+1$, that is, 6; and so on generally, for any number whatever.

Now let . . . b, c, d, etc., each $=a$;

Then, $A=a+a+a+a+$, etc., to n terms $=na$.

$B=a^2+a^2+$, etc., $=a^2$ taken as many times as is equal to the No. of combinations of n things taken *two* together, which is (Art. 308), $\Bigg\} = \dfrac{n(n-1)a^2}{1 \cdot 2}$

$C=a^3+a^3+$, etc., $=a^3$ taken as many times as is equal to the No. of combinations of the things taken *three* together, which is (Art. 308), $\Bigg\} = \dfrac{n(n-1)(n-2)a^3}{1 \cdot 2 \cdot 3}$.

Etc. = etc.

$K=aaa$ to n factors $=a^n$

Also, $(x+a)(x+b)(x+c)\ \ldots\ldots\ldots\ (x+l)=(x+a)^n$.

$$\therefore\ (x+a)^n=x^n+nax^{n-1}+\frac{n(n-1)}{1\cdot 2}a^2x^{n-2}+\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}a^3x^{n-3}$$
$$+\qquad \ldots +a^n.$$

By changing x to a and a to x, we have

$$(a+x)^n=a^n+na^{n-1}x+\frac{n(n-1)}{1\cdot 2}a^{n-2}x^2+$$
$$\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}a^{n-3}x^3\ \ldots\ldots\ldots\ +x^n$$

Let $a=1$; then, since every power of 1 is 1,

$$(1+x)^n=1+nx+\frac{n(n-1)}{1\cdot 2}x^2+\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}x^3+\ \ldots\ldots\ +x^n.$$

Corollary 1.—The sum of the exponents of a and x in each term $=n$.

Corollary 2.—If either term of the binomial is negative, every *odd* power of that term will be negative (Art. 193); therefore, the terms which contain the odd powers will be *negative*.

$$\therefore\ (1-x)^n=1-nx+\frac{n(n-1)}{1\cdot 2}x^2-\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}x^3+, \text{ etc.}$$

Corollary 3.—Since the last factors, in the fraction which forms the coëfficient, are for the 2d term $\frac{n}{1}$, for the 3d term $\frac{n-1}{2}$, for the 4th term $\frac{n-2}{3}$, etc.; therefore, for the r^{th} term they will be $\frac{n-(r-2)}{r-1}$

Also, for the exponents of a and x, we have in the 2d term $a^{n-1}x$, in the 3d term $a^{n-2}x^2$, in the 4th term $a^{n-3}x^3$; therefore, in the r^{th} term, we shall have $a^{n-(r-1)}x^{r-1}$.

Hence, the general term of the series is

$$\frac{n(n-1)(n-2)\ \ldots\ldots\ (n-r+2)}{1\cdot 2\cdot 3\ \ldots\ \ \ldots\ (r-1)}a^{n-r+1}x^{r-1}.$$

This is called the *general term*, because by making $r=2$, 3, 4, etc., all the others can be deduced from it.

EXAMPLE.—Required the 5^{th} term of $(a-x)^7$.

Here, $r=5$, and $n=7$; therefore, the term required

$$=\frac{4\,.\,5\,.\,6\,.\,7}{1\cdot2\cdot3\cdot4}(a)^3(-x)^4=35a^3x^4.$$

Corollary 4.—If n be a positive integer, and $r=n+2$; then, $(n-r+2)$ becomes 0, and the $(n+2)$ term vanishes; therefore, the series consists of $(n+1)$ terms altogether; that is,

The number of terms is one greater than the exponent of the power to which the binomial is to be raised.

Corollary 5.—When the index of the binomial is a positive integer, the coëfficients of the terms taken in an inverse order from the end of the series, are equal to the coëfficients of the corresponding terms taken in a direct order from the beginning.

If we compare the expansion of $(a+x)^n$, and $(x+a)^n$, we have

$$(a+x)^n=a^n+na^{n-1}x+\frac{n(n-1)}{1\cdot2}a^{n-2}x^2+\frac{n(n-1)(n-2)}{1\cdot2\cdot3}a^{n-3}x^3+\text{, etc.}$$

$$(x+a)^n=x^n+nx^{n-1}a+\frac{n(n-1)}{1\cdot2}x^{n-2}a^2+\frac{n(n-1)(n-2)}{1\cdot2\cdot3}x^{n-3}a^3+\text{, etc.}$$

Since the binomials are the same, the series resulting from their expansion must be the same, except that the order of the terms will be inverted. It is clearly seen that the coëfficients of the corresponding terms are equal.

Hence, in expanding such a binomial, the latter half of the expansion may be taken from the first half.

EXAMPLE.—Expand $(a-b)^5$

Here the number of terms $(n+1)$ is 6; therefore, it is only necessary to find the coëfficients of the first three, thus:

$$(a-b)^5=a^5-5a^4b+\frac{5\cdot4}{1\cdot2}a^3b^2-10a^2b^3+5ab^4-b^5.$$

Corollary 6.—The sum of the coëfficients, where both terms are positive, is always equal to 2^n.

For if we make $x=a=1$; then, . $(x+a)^n=(1+1)^n=2^n$.

311. From an inspection of the general expansion of $(a+x)^n$, it is evident that

If the coëfficient of any term be multiplied by the exponent of the first letter of the binomial in that term, and the product be divided by the number of the term, the quotient will be the coëfficient of the next term.

For examples, see *Newton's Theorem*, Art. 172.

312. To expand a binomial affected with coëfficients or exponents, as $(2a^2-3b^3)^4$, see *Newton's Theorem*, Art. 172.

313. By means of the Binomial Theorem, we can raise any polynomial to any power. Thus, let it be required to raise $a-b+c$ to the third power.

Let $a-b=m$, etc., as already explained, Art. 172.

1. Expand $(a+b)^8$, $(a-b)^7$, and $(5-4x)^4$.

(1) Ans. $a^8+8a^7b+28a^6b^2+56a^5b^3+70a^4b^4+56a^3b^5$
$+28a^2b^6+8ab^7+b^8$.

(2) Ans. $a^7-7a^6b+21a^5b^2-35a^4b^3+35a^3b^4-21a^2b^5$
$+7ab^6-b^7$.

(3) Ans. $625-2000x+2400x^2-1280x^3+256x^4$.

2. Required the coëfficient of x^6 in the expansion of $(x+y)^{10}$. Ans. 210.

3. Find the 5^{th} term of the expansion of $(c^2-d^2)^{12}$.
Ans. $495c^{16}d^8$.

Suggestion.—(See Cor. 3, Art. 310.) Instead of a, x, n, and r, substitute c^2, $-d^2$, 12, and 5.

4. Find the 7^{th} term of $(a^3+3ab)^9$. Ans. $61236a^{15}b^6$.

5. Find the middle term of $(a^m+x^n)^{12}$. A. $924a^{6m}x^{6n}$.

6. Find the 8^{th} term of $(1+x)^{11}$. Ans. $330x^7$.

7. Expand $(3ac-2bd)^5$. Ans. $243a^5c^5-810a^4c^4bd+1080a^3c^3b^2d^2-720a^2c^2b^3d^3+240acb^4d^4-32b^5d^5$.

8. Expand $(a+2b-c)^3$. Ans $a^3+6a^2b+12ab^2+8b^3-3a^2c-12abc-12b^2c+3ac^2+6bc^2-c^3$.

9. Prove that the sum of the coëfficients of the odd terms of $(a+x)^n$, is equal to the sum of the coëfficients of the even terms.

X. INDETERMINATE COEFFICIENTS: BINOMIAL THEOREM, GENERAL DEMONSTRATION: SUMMATION AND INTERPOLATION OF SERIES.

314. Indeterminate Coefficients.—The method of developing algebraic expressions, by assuming a series with unknown coëfficients, and finding the values of the assumed coëfficients, is termed the method of *Indeterminate Coëfficients.* It depends on the following

THEOREM.

If $A+Bx+Cx^2+Dx^3+$, etc., $=A'+B'x+C'x^2+D'x^3+$, etc., for every possible value of x (A, B, A′, B′, etc., not containing x, and x being a variable quantity) we shall have $A=A'$, $B=B'$, $C=C'$, etc.; that is,

The coëfficients of the terms involving the same powers of x in the two series, are respectively equal.

For, by transposing all the terms into the first member, we have $A-A'+(B-B')x+(C-C')x^2+(D-D')x^3+$, etc., $=0$.

If $A-A'$ is not equal to 0, let it be equal to some quantity p; then, we have $(B-B')x+(C-C')x^2+(D-D')x^3+$, etc., $=-p$.

Now, since A and A' are constant quantities, their difference, p, must be constant; but $-p=(B-B')x+(C-C')x^2+$, etc., a quantity which may evidently have various values, since it depends upon x; therefore, the same quantity (p) is both *fixed* and *variable*, which is *impossible*.

Hence, there is no *possible quantity* (p) which can express the difference $A-A'$; or, in other words,

$$A-A'=0 \therefore A=A'.$$

Hence, $(B-B')x+(C-C')x^2+(D-D')x^3+$, etc., $=0$.

By dividing each side by x, we have

$$B-B'+(C-C')x+(D-D')x^2+, \text{ etc.}, =0.$$

Reasoning as before, we may show that $B=B'$; and so on, for the remaining coëfficients of the like powers of x.

Corollary.—If we have an equation of the form

$$A+Bx+Cx^2+Dx^3+Ex^4+, \text{ etc.}, =0,$$

which is true for *any value whatever of* x; then, $A=0$, $B=0$, $C=0$, etc.; that is, *each coëfficient is separately equal to zero.*

For the right hand member may evidently be put under the form $0+0x+0x^2+0x^3+$, etc.; then, comparing the coëfficients of the like powers of x, we have $A=0$, $B=0$, $C=0$, etc.

315. Let it be required to develope $\frac{a}{a+bx}$ into a series without a resort to division.

The series will consist of the powers of x multiplied by certain undetermined coëfficients, depending on either a or b, or both of them, and x will not enter into the first term; therefore, assume

$$\frac{a}{a+bx}=A+Bx+Cx^2+Dx^3+, \text{ etc.}$$

Multiply both sides by the denominator $a+bx$, and arrange the terms according to the powers of x; we thus obtain

$$a=Aa+\left.\begin{matrix}Ba\\+Ab\end{matrix}\right| x+\left.\begin{matrix}Ca\\+Bb\end{matrix}\right| x^2+\left.\begin{matrix}Da\\+Cb\end{matrix}\right| x^3+, \text{ etc.}$$

But by the preceding theorem and corollary,

$$a=\text{A}a;\ \text{hence, } \text{A}=1;$$
$$\text{B}a+\text{A}b=0;\quad \text{"}\quad \text{B}=-\frac{b}{a};$$
$$\text{C}a+\text{B}b=0;\quad \text{"}\quad \text{C}=+\frac{b^2}{a^2};$$
$$\text{D}a+\text{C}b=0;\quad \text{"}\quad \text{D}=-\frac{b^3}{a^3}, \text{ etc.}$$

Substituting these values in the assumed series, we find

$\dfrac{a}{a+bx}=1-\dfrac{b}{a}x+\dfrac{b^2}{a^2}x^2-\dfrac{b^3}{a^3}x^3+\dfrac{b^4}{a^4}x^4$, etc., the same as would be obtained by actual division.

316. A series with indeterminate coëfficients is generally assumed to proceed according to the ascending integral and positive powers of x, beginning with x^0; but in many series this is not the case. The error in the assumption will then be shown, either by an impossible result, or by finding the coëfficients of the terms which do not exist in the *actual* series, equal to zero.

Thus, if it be required to develope $\dfrac{1}{3x-x^2}$, and we assume the series to be $\text{A}+\text{B}x+\text{C}x^2+\text{D}x^3+\text{E}x^4+$, etc., we have, after clearing of fractions,

$$1=3\text{A}x+(3\text{B}-\text{A})x^2+(3\text{C}-\text{B})x^3+, \text{ etc.};$$

from which, by equating the coëfficients of the same powers of x,

$$1=0,\ 3\text{A}=0, \text{ etc.}$$

The first equation, $1=0$, being absurd, we infer that the expression can not be developed under the assumed form. But,

$\dfrac{1}{3x-x^2}=\dfrac{1}{x}\times\dfrac{1}{3-x}$. Putting $\dfrac{1}{3-x}=\text{A}+\text{B}x+\text{C}x^2+$, etc.,

clearing of fractions, and equating the coëfficients of the like powers of x, we find $\text{A}=\frac{1}{3}$, $\text{B}=\frac{1}{9}$, $\text{C}=\frac{1}{27}$, $\text{D}=\frac{1}{81}$, etc. Hence,

$$\frac{1}{3x-x^2}=\frac{1}{x}\left(\frac{1}{3}+\frac{x}{9}+\frac{x^2}{27}+\frac{x^3}{81}+, \text{ etc. }\right)=\frac{x^{-1}}{3}+\frac{x^0}{9}+\frac{x}{27}+\frac{x^2}{81}+, \text{ etc.}$$

Or, since the division of 1 by the first term of the denominator gives $\frac{1}{3x}$, or $3x^{-1}$, we ought to have assumed

$$\frac{1}{3x-x^2}=Ax^{-1}+B+Cx+Dx^2+, \text{ etc.}$$

Again, if we assume $\frac{1-x^2}{1+x^2-x^4}=A+Bx+Cx^2+Dx^3+$, etc.; we shall find the true series to be $1-2x^2+3x^4-5x^6+$, etc., the coëfficients B, D, F, etc., becoming zero.

317. Evolution by indeterminate coëfficients.

EXAMPLE.—Extract the square root of a^2+x^2.

Assume $\sqrt{(a^2+x^2)}=A+Bx+Cx^2+Dx^3+Ex^4+$, etc.

Squaring both sides, we have,

$$a^2+x^2=A^2+2ABx+\left.\begin{array}{r}2AC\\+B^2\\ \end{array}\right| x^2+\left.\begin{array}{r}2AD\\+2BC\\ \end{array}\right| x^3+\left.\begin{array}{r}2AE\\+2BD\\+C^2\end{array}\right| x^4+, \text{ etc.}$$

$\therefore\ A^2=a^2,\ 2AB=0,\ 2AC+B^2=1,\ 2AD+2BC=0$, etc.

And, $A=a,\ B=0,\ C=\frac{1}{2a},\ D=0,\ E=-\frac{1}{8a^3}$, etc.

Therefore, $\sqrt{(a^2+x^2)}=a+\frac{x^2}{2a}-\frac{x^4}{8a^3}+$, etc

318. Decomposition of Rational Fractions.—Fractions whose denominators can be separated into certain factors, may often be decomposed into other fractions whose denominators shall consist of one or more of these factors. To illustrate by an example.

Decompose $\frac{5x-14}{x^2-6x+8}$ into two other fractions whose denominators shall be the factors of x^2-6x+8.

Since $x^2-6x+8=(x-2)(x-4)$, (Art. 234), assume

$$\frac{5x-14}{x^2-6x+8}=\frac{A}{x-2}+\frac{B}{x-4}.$$

Reducing the fractions to a common denominator,

We have $$\frac{5x-14}{x^2-6x+8}=\frac{A(x-4)+B(x-2)}{(x-2)(x-4)},$$

Or, $$5x-14=A(x-4)+B(x-2)=(A+B)x-4A-2B.$$

Now, since this equation is true for any value whatever of x, we may equate the coëfficients (Art. 314); this gives

$A+B=5$; $-4A-2B=-14$; whence, $A=2$, and $B=3$.

And $$\frac{5x-14}{x^2-6x+8}=\frac{2}{x-2}+\frac{3}{x-4}.$$

By the method of Indeterminate Coëfficients, show that

1. $\dfrac{1+2x}{1-3x}=1+5x+15x^2+45x^3+$, etc.

2. $\dfrac{1+2x}{1-x-x^2}=1+3x+4x^2+7x^3+11x^4+18x^5+$, etc.

3. $\dfrac{1+x}{(1-x)^3}=1^2+2^2x+3^2x^2+4^2x^3+5^2x^4+$, etc.

4. $\sqrt{1-x}=1-\dfrac{x}{2}-\dfrac{x^2}{2\cdot4}-\dfrac{3x^3}{2\cdot4\cdot6}-\dfrac{3\cdot5x^4}{2\cdot4\cdot6\cdot8}-$, etc.

5. $\sqrt{(1+x+x^2)}=1+\dfrac{x}{2}+\dfrac{3x^2}{8}-\dfrac{3x^3}{16}+$, etc.

6. $\dfrac{1+x}{x-x^2}=\dfrac{1}{x}+\dfrac{2}{1-x}$.

7. $\dfrac{x+1}{x^2-7x+12}=\dfrac{5}{x-4}-\dfrac{4}{x-3}$.

8. $\dfrac{x^2}{(x^2-1)(x-2)}=\dfrac{4}{3(x-2)}-\dfrac{1}{2(x-1)}+\dfrac{1}{6(x+1)}$.

BINOMIAL THEOREM,

WHEN THE EXPONENT IS FRACTIONAL OR NEGATIVE.

319. We shall now proceed to prove the truth of the Binomial Theorem generally; that is, to show that

$$(a+b)^n=a^n+na^{n-1}b+\frac{n(n-1)}{1\cdot 2}a^{n-2}b^2+\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}a^{n-3}b^3+\text{, etc.}$$

whether n be integral or fractional, positive or negative.

Since . . . . . . $a+b=a\left(1+\frac{b}{a}\right);$

Therefore, $(a+b)^n=a^n\left(1+\frac{b}{a}\right)^n=a^n(1+x)^n$, if $x=\frac{b}{a}$.

Hence, if we can find the law of the expansion of $(1+x)^n$ we may obtain that of $(a+b)^n$, by writing $\frac{b}{a}$ for x, and multiplying by a^n. We shall therefore prove that, in all cases,

$$(1+x)^n=1+nx+\frac{n(n-1)}{1\cdot 2}x^2+\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}x^3+\text{, etc.}$$

The proof may be divided into two parts:

1st. To show that $(1+x)^n=1+nx+$, etc.
2d. To find the general law of the coëfficients.

First.—To prove that the coëfficient of the second term of the expansion of $(1+x)^n$ is n, whether n be integral or fractional, positive or negative.

Let the index be positive and integral; then, since by multiplication we know that

$$(1+x)^2=1+2x+\text{, etc.,}$$
$$(1+x)^3=1+3x+\text{, etc.;}$$

Let us assume that $(1+x)^{n-1}=1+(n-1)x+$, etc.

Multiply both sides of this equality by $1+x$; then,

$$(1+x)^{n-1}(1+x)=\{1+(n-1)x+\text{, etc. }\}(1+x);$$

Or, $(1+x)^n=1+nx+$, etc., by multiplication.

Hence, if the proposition is true for any one index $n-1$, it will be true for the next higher index n. Now, by multiplication, it is true for the index 3, it is therefore true for the index $3+1=4$; for the index $4+1=5$, and so on. Hence, by continued induction, it is always true for n when it is integral and positive.

Next, let n be a fraction $=\frac{p}{q}$.

Also, let $(1+x)^{\frac{p}{q}}=1+ax+$, etc., $=1+\mathrm{A}x$, where $\mathrm{A}x$ is put to represent all the terms after the first.

Since . . . . $(1+x)^{\frac{p}{q}}=1+\mathrm{A}x$, $\therefore$ by raising both sides to the q power, . $(1+x)^p=(1+\mathrm{A}x)^q$;

$$\therefore 1+px+, \text{ etc.}, =1+q\mathrm{A}x+, \text{ etc.},$$
$$=1+q(ax+, \text{ etc.})+, \text{ etc.},$$
$$=1+qax+, \text{ etc.}$$

By equating the coëfficients of the like powers of x (Art. 314),

$$p=qa \;\therefore\; a=\frac{p}{q}, \text{ and } (1+x)^{\frac{p}{q}}=1+\frac{p}{q}x+, \text{ etc.}$$

Lastly, let n be negative; then, (Art. 81),

$$(1+x)^{-n}=\frac{1}{(1+x)^n}=\frac{1}{1+nx+, \text{ etc.}}=1-nx+, \text{ etc., by division.}$$

Therefore, $(1+x)^n=1+nx+$, etc., whatever be the value of n.

$$\text{Therefore, } (a+b)^n=a^n\left(1+\frac{b}{a}\right)^n=a^n(1+n\frac{b}{a}+, \text{ etc.}),$$
$$=a^n+na^{n-1}b+, \text{ etc.},$$

and the first two terms of the series are determined.

SECOND.—To find the general law of the coëfficients.

Let $(1+x)^n=1+nx+\mathrm{B}x^2+\mathrm{C}x^3+\mathrm{D}x^4+$, etc., where B, C, D, etc., depend upon n.

For x, put $x+z$, and consider $(x+z)$ as *one term*; then,

$$\{1+(x+z)\}^n=1+n(x+z)+\mathrm{B}(x+z)^2+\mathrm{C}(x+z)^3+, \text{ etc.}$$

But . $(a+b)^n=a^n+na^{n-1}b+$, etc.;

$$\therefore (x+z)^2=x^2+2xz+, \text{ etc.};$$
$$(x+z)^3=x^3+3x^2z+, \text{ etc.};$$
$$(x+z)^4=x^4+4x^3z+, \text{ etc.};$$
$$\therefore \{1+(x+z)\}^n=1+nx+\mathrm{B}x^2+\mathrm{C}x^3+\mathrm{D}x^4+, \text{ etc.},$$
$$+(n+2\mathrm{B}x+3\mathrm{C}x^2+4\mathrm{D}x^3+, \text{ etc.})z+, \text{ etc.},$$
$$=(1+x)^n+(n+2\mathrm{B}x+3\mathrm{C}x^2+4\mathrm{D}x^3+, \text{ etc.})z+, \text{ etc., (A).}$$

But, considering $(1+x)$ as *one term*, $(1+x+z)^n=\{(1+x)+z\}^n$; and $\{(1+x)+z\}^n=(1+x)^n+n(1+x)^{n-1}z+$, etc. (B).

Equating the coefficients of z in (A) and (B),

$$n+2Bx+3Cx^2+4Dx^3+\text{, etc.,} =n(1+x)^{n-1}.$$

Multiplying both sides by $1+x$, we have

$$\left.\begin{array}{l} n+2Bx+3Cx^2+4Dx^3+\text{, etc.} \\ +\ nx+2Bx^2+3Cx^3+\text{, etc.} \end{array}\right\} \begin{array}{l} =n(1+x)^n \\ =n(1+nx+Bx^2+Cx^3+\text{, etc.}) \end{array}$$

By equating the coefficients of the same powers of x, we have

$$2B+n=n^2 \;.\,.\; 2B=n^2-n=n(n-1).$$

$$B=\frac{n(n-1)}{1\cdot 2};$$

$$3C+2B=Bn \;\therefore\; 3C=B(n-2);$$

$$C=\frac{B(n-2)}{3}=\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3};$$

Also, $4D+3C=nC \;\therefore\; 4D=C(n-3)$;

$D=\frac{C(n-3)}{4}=\frac{n(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3\cdot 4}$; and so on for E, F, G, etc.

$\therefore\ (1+x)^n=1+nx+\frac{n(n-1)}{1\cdot 2}x^2+\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}x^3+$, etc., and

$\therefore$ putting $\frac{b}{a}$ for x, $(a+b)^n=a^n\left(1+\frac{b}{a}\right)^n$,

$$=a^n\{1+n\frac{b}{a}+\frac{n(n-1)b^2}{1\cdot 2\ a^2}+\frac{n(n-1)(n-2)b^3}{1\cdot 2\cdot 3\quad a^3}+\text{, etc.,}\},$$

$$=a^n+na^{n-1}b+\frac{n(n-1)}{1\cdot 2}a^{n-2}b^2+\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}a^{n-3}b^3+\text{, etc.}$$

If $-b$ be put for b, then since the *odd* powers of $-b$ are ***negative*** (Art. 193) and the *even* powers positive,

$$(a-b)^n=a^n-na^{n-1}b+\frac{n(n-1)}{1\cdot 2}a^{n-2}b^2-\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}a^{n-3}b^3+,$$

etc., which establishes the Binomial Theorem in its most general form.

REMARK.—From the preceding, corollaries, similar to those in Art. 310, may be drawn, but it is not necessary to repeat them. The following additional proposition is sometimes useful.

320. To find the *numerically* greatest term in the expansion of $(a+b)^n$.

I. If n is a positive integer:

From Cor. 3, Art. 310, it appears that the $(r+1)^{th}$ term may be formed from the r^{th} by multiplying the latter by $\frac{n-r+1}{r}\cdot\frac{b}{a}$, or $\left(\frac{n+1}{r}-1\right)\frac{b}{a}$, and this multiplier diminishes as r increases; while it is greater than 1, each term is greater than the preceding; and the value of r, which first makes it less than 1, indicates the greatest term; that is, the r^{th} term will be the greatest when $\left(\frac{n+1}{r}-1\right)\frac{b}{a}$ is first <1 or r first $>\frac{(n+1)b}{a+b}$.

From the nature of the case, r is necessarily integral; if $\frac{(n+1)b}{a+b}$ is fractional, take $r=$ the *first* integer $>\frac{(n+1)b}{a+b}$, and the r^{th} term will be the greatest. If $\frac{(n+1)b}{a+b}$ is an integer, and we take $r=\frac{(n+1)b}{a+b}$, then the r^{th} term $=$ the $(r+1)^{th}$, and each of these is greater than any other term; for this can only occur when $\frac{n-r+1}{r}\cdot\frac{b}{a}=1$.

II. If n is a positive fraction:

If $\frac{b}{a}>1$ there is no greatest term, for the series will evidently *diverge*. But if $\frac{b}{a}<1$ the series will have its greatest term (or terms) whose position may be ascertained as in I.

III. If n be negative, either an integer or a fraction >1:

The multiplier that changes the r^{th} term into the $(r+1)^{th}$, viz., $\frac{-n-r+1}{r}\cdot\frac{b}{a}$ may be written $-\left(\frac{n+r-1}{r}\right)\frac{b}{a}$, and as the *numerically* greatest term is sought, disregard the sign of the multiplier: then, as in I, the r^{th} term will be the greatest when $\frac{n+r-1}{r}\cdot\frac{b}{a}$ is first <1 or r first $>\frac{b(n-1)}{a-b}$.

As in I, if $\frac{b(n-1)}{a-b}$ be a whole number there are two equal terms each greater than any other; and, as in II, if $\frac{b}{a}$ be >1, there is no greatest term.

IV. If n be negative and < 1, and $\frac{b}{a} < 1$, the first term is the greatest; for in this case the multiplier $\frac{n+r-1}{r} \cdot \frac{b}{a}$ is < 1 for *all* values of r, that is, each term is less than the preceding.

NOTE.—If b is negative, since it is the numerical value of the term that is to be considered, we may disregard the sign of b and apply the appropriate one of the preceding rules. [Cf. Todhunter's Algebra, Art. 520.]

EXAMPLES.—Find the greatest term in each of the following expansions:

1. $(2+\frac{5}{3})^6$. Here $\frac{(n+1)b}{a+b} = \frac{35}{11}$ $\therefore$ $r=4$ gives the greatest term $= \frac{6.5.4}{1.2.3} \cdot 2^3 \cdot \frac{5^3}{3^3} = \frac{20000}{81}$.

2. $(1+\frac{3}{4})^{\frac{10}{3}}$. Ans. 2^d.

3. $(\frac{2}{3}+\frac{5}{3})^6$. Ans. 5^{th} and 6^{th}.

4. $(1+\frac{1}{5})^{-12}$. Here $\frac{(n-1)b}{a-b} = \frac{11}{4}$ $\therefore$ $r=3$ gives the greatest term $= \frac{12 \cdot 13}{1 \cdot 2} \cdot \frac{1}{5^2} = \frac{78}{25}$.

5. $(1+\frac{5}{7})^{-3}$. Here $\frac{(n-1)b}{a-b} = 5$ $\therefore$ $5^{th} = 6^{th}$, and each is greater than any other term.

6. $(1-\frac{7}{12})^{-\frac{8}{3}}$. Ans. 3^{rd}.

321. In the application of the Binomial Theorem, it is merely necessary to take the general formula $(a+b)^n = a^n + na^{n-1}b +$, etc., and substitute the given quantities in the formula, and then reduce each term to its most simple form

EXAMPLE.—1. Find the expansion of $(1+x)^{\frac{1}{2}}$.

Here, $a=1$, $b=x$, $n=\frac{1}{2}$.

$$\therefore (1+x)^{\frac{1}{2}} = 1 + \tfrac{1}{2}x + \frac{\frac{1}{2}(\frac{1}{2}-1)}{1 \cdot 2}x^2 + \frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)}{1 \cdot 2 \cdot 3}x^3 +, \text{ etc.}$$

$$= 1 + \tfrac{1}{2}x - \frac{1 \cdot 1}{2 \cdot 4}x^2 + \frac{1 \cdot 1 \cdot 3}{2 \cdot 4 \cdot 6}x^3 - \frac{1 \cdot 1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 8}x^4 +, \text{ etc.}$$

EXAMPLE.—2. Develope $(1-x)^{-\frac{1}{2}}$.

Here, $a=1$, $b=-x$, $n=-\frac{1}{2}$.

$\therefore (1-x)^{-\frac{1}{2}}=1-\frac{1}{2}(-x)+\frac{(-\frac{1}{2})(-\frac{1}{2}-1)}{1\cdot 2}(-x)^2+$, etc.,

$=1+\frac{1}{2}x+\frac{1\cdot 3}{2\cdot 4}x^2+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}x^3+$, etc.

As the general formula $(1\pm x)^n=1\pm nx+\frac{n(n-1)}{1\cdot 2}x^2\pm$, etc., is more easily retained in the memory, and is less complicated, it is generally most convenient to reduce the quantity to be expanded to this form. Thus:

Develope $\sqrt{a+b}$ into a series.

Since $a+b=a\left(1+\frac{b}{a}\right)$, $\therefore \sqrt{a+b}=\sqrt{a}\left(1+\frac{b}{a}\right)^{\frac{1}{2}}$.

Here, $x=\frac{b}{a}$, $n=\frac{1}{2}$; and since

$(1+x)^n=1+nx+\frac{n(n-1)}{1\cdot 2}x^2+\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}x^3+$, etc.,

$\therefore\left(1+\frac{b}{a}\right)^{\frac{1}{2}}=1+\frac{1}{2}\frac{b}{a}+\frac{\frac{1}{2}(\frac{1}{2}-1)}{1\cdot 2}\cdot\frac{b^2}{a^2}+\frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)}{1\cdot 2\cdot 3}\cdot\frac{b^3}{a^3}+$, etc.,

$=1+\frac{1}{2}\frac{b}{a}-\frac{1}{2\cdot 4}\cdot\frac{b^2}{a^2}+\frac{1\cdot 3}{2\cdot 4\cdot 6}\cdot\frac{b^3}{a^3}-$, etc.

Hence, $\sqrt{a+b}=\sqrt{a}\left(1+\frac{b}{2a}-\frac{b^2}{8a^2}+\frac{b^3}{16a^3}-\frac{5b^4}{128a^4}+\text{, etc.}\right)$.

1. $\frac{1}{1-x}=(1-x)^{-1}=1+x+x^2+x^3+x^4+$, etc.

2. $\frac{1}{(1-x)^2}=(1-x)^{-2}=1+2x+3x^2+4x^3+5x^4+$, etc.

3. $\frac{a^2}{(a+x)^2}=a^2(a+x)^{-2}=a^2\times a^{-2}\left(1+\frac{x}{a}\right)^{-2}=\left(1+\frac{x}{a}\right)^{-2}$

$$=1-\frac{2x}{a}+\frac{3x^2}{a^2}-\frac{4x^3}{a^3}+\frac{5x^4}{a^4}-\text{, etc.}$$

4. $\sqrt[3]{1-x^3}=1-\frac{x^3}{3}-\frac{x^6}{9}-\frac{5x^9}{81}-$, etc.

5. $\sqrt{a^2+x}=a+\frac{x}{2a}-\frac{x^2}{8a^3}+\frac{x^3}{16a^5}-\frac{5x^4}{128a^7}+$, etc.

6. $(a^3-x)^{\frac{1}{3}}=a-\frac{x}{3a^2}-\frac{x^2}{9a^5}-\frac{5x^3}{81a^8}-\frac{10x^4}{243a^{11}}-$, etc.

7. $(1+2x)^{\frac{1}{2}}=1+x-\frac{1}{2}x^2+\frac{1}{2}x^3-\frac{5}{8}x^4+$, etc.

8. $\sqrt{a^2-x^2}=a-\frac{x^2}{2a}-\frac{x^4}{8a^3}-\frac{x^6}{16a^5}-\frac{5x^8}{128a^7}-$, etc.

9. $\sqrt[3]{a+x}=\sqrt[3]{a}(1+\frac{1x}{3a}-\frac{1x^2}{9a^2}+\frac{5x^3}{81a^3}-\frac{10x^4}{243a^4}+$, etc.).

10. $(a^3+x^3)^{\frac{1}{3}}=a(1+\frac{x^3}{3a^3}-\frac{2x^6}{3\cdot 6a^6}+\frac{2\cdot 5x^9}{3\cdot 6\cdot 9a^9}-$, etc.).

11. $\sqrt[3]{9}=\sqrt[3]{8+1}=2+\frac{2}{3}\cdot\frac{1}{8}-\frac{2}{9}\cdot\frac{1}{8^2}+\frac{10}{81}\cdot\frac{1}{8^3}-$, etc.

12. $(a^3-x^3)^{\frac{1}{3}}=a(1-\frac{x^3}{3a^3}-\frac{2x^6}{3\cdot 6a^6}-\frac{2\cdot 5x^9}{3\cdot 6\cdot 9a^9}-$, etc.).

13. $\frac{a^3}{(a^3-x^3)^{\frac{2}{3}}}=a+\frac{2x^3}{3a^2}+\frac{2\cdot 5x^6}{3\cdot 6a^5}+\frac{2\cdot 5\cdot 8x^9}{3\cdot 6\cdot 9a^8}+$, etc.

Here, $\frac{a^3}{(a^3-x^3)^{\frac{2}{3}}}=a^3(a^3-x^3)^{-\frac{2}{3}}=a^3\times(a^3)^{-\frac{2}{3}}\left(1-\frac{x^3}{a^3}\right)^{-\frac{2}{3}}=a^3$
$\times a^{-2}\left(1-\frac{x^3}{a^3}\right)^{-\frac{2}{3}}=a\left(1-\frac{x^3}{a^3}\right)^{-\frac{2}{3}}$.

322. To find the approximate roots of numbers by the Binomial Theorem.

Let N represent any proposed number whose n^{th} root is required; take a such that a^n is the nearest perfect n^{th} power

to N, so that $N=a^n\pm b$, b being small compared with a, and $+$ or $-$, according as $N>$ or $<a^n$;

Then, $\sqrt[n]{N}=a\left(1\pm\frac{b}{a^n}\right)^{\frac{1}{n}}=$, by writing $\frac{b}{a^n}$ for b in the general formula;

$$a\{1\pm\frac{1}{n}\cdot\frac{b}{a^n}-\frac{1}{n}\cdot\frac{n-1}{2n}\left(\frac{b}{a^n}\right)^2\pm\frac{1}{n}\cdot\frac{n-1}{2n}\cdot\frac{2n-1}{3n}\left(\frac{b}{a^n}\right)^3-\text{, etc.}\}.$$

Of this series a few terms only, when b is small with regard to a^n, will give the required root to a considerable degree of accuracy.

14. Required the approximate cube root of 128.

Here, $\sqrt[3]{128}=\sqrt[3]{5^3+3}=5\sqrt[3]{1+\frac{3}{125}}$;

$$=5\{1+\frac{1}{3}\cdot\frac{3}{125}-\frac{1}{3}\cdot\frac{1}{3}\left(\frac{3}{125}\right)^2+\frac{1}{3^2}\cdot\frac{5}{9}\left(\frac{3}{125}\right)^3-, \ldots\};$$

$$=5+\frac{1}{5^2}-\frac{1}{5^5}+\frac{1}{3}\cdot\frac{1}{5^7}-\ldots=5+\frac{2^2}{10^2}-\frac{2^5}{10^5}+\frac{1}{3}\cdot\frac{2^7}{10^7}-\ldots$$

$$=5+0.04-0.00032+0.0000042-\ldots$$

$$=5.0396842.$$

323. In the preceding example, we obtain only an approximate value. To determine the *limit* in the error occasioned by neglecting the remaining terms of the series, let R be the true root, and as the terms are alternately positive and negative, put

$R\ =a-b+c-d+e-f+g-h+k-l+$, etc., and let
$R'=a-b+c-d+e-f$,
$R''=a-b+c-d+e-f+g$.

Then, since the terms continually decrease, $a-b$, $c-d$, $e-f$, $g-h$, etc., are all positive, and therefore R', which contains three only of those differences, will be *less* than R. For the same reason, all the pairs of terms after g, as $-h+k$, $-l+m$, etc., will be all

negative, and R″ will be *greater* than R; therefore, the true value of the series lies between R′ and R″, or between

$$a-b+c-d+e-f, \text{ and}$$
$$a-b+c-d+e-f+g. \text{ Hence,}$$

The error committed by the omission of any number of the terms of a converging series whose signs are alternately positive and negative, is less than the first omitted term.

Thus, in the preceding example, had we stopped at the 3d term, the error would have been less than .0000042.

15. Find the 5^{th} root of 35. Ans. 2.036172+

Here, . . . $N=35=32+3=2^5\left(1+\frac{3}{2^5}\right)$.

16. The student may solve the following examples:

(1). $\sqrt{10}=\sqrt{9+1}=3.16227$ . . . true to 0.00001.
(2). $\sqrt[3]{24}=\sqrt[3]{27-3}=2.88449$ . . . true to 0.00001.
(3) $\sqrt[7]{108}=\sqrt[7]{128-20}=1.95204$ . . . true to 0.00001.

REMARK.—The *n*th root may be extracted by the formula in Art. 321, the number whose root is to be extracted being divided into any two parts whatever. The advantage of the formula in Art. 322 consists in the rapid convergence of its terms.

THE DIFFERENTIAL METHOD OF SERIES.

324. The **Differential Method** is used, 1st, to find the successive differences of the terms of a series; 2d, to find any particular term of the series; or, 3d, to find the sum of a finite number of its terms.

If, in any series, we take the first term from the second, the second from the third, the third from the fourth, and so on, the new series thus formed is called the *first order of differences.*

If we proceed with this new series in the same manner, we shall obtain another series, termed the *second order of differences.*

In a similar manner we find the *third*, *fourth*, etc., orders of differences.

If we have the series,	1 , 8 , 27 , 64 , 125 , 216, . . .
The 1st order of differences is	7 , 19 , 37 , 61 , 91 , ...
The 2d " " " "	12 , 18 , 24 , 30 , ...
The 3d " " " "	, 6 , 6 , 6 , ...

325. Problem I.—*To find the first term of any order of differences.*

Let the series be $a, b, c, d, e, \ldots\ldots$; then, the respective orders of differences are,

1st order, $b-a \quad , \quad c-b \quad , \quad d-c \quad , \quad e-d, \ldots$

2d order, $c-2b+a \ , \ d-2c+b \ , \ e-2d+c, \ldots$

3d order, $d-3c+3b-a, \ e-3d+3c-b, \ldots$

4th order, $e-4d+6c-4b+a.$

Here, each difference pointed off by commas, though a compound quantity, is called a *term*. Thus, the first term in the 1st order, is $b-a$; in the second order, $c-2b+a$, etc.

If we denote the first terms in the 1st, 2d, 3d, 4th, etc., orders of differences by D_1, D_2, D_3, D_4, etc., and invert the order of the letters, we have

$$D_1=-a+b;\ D_2=a-2b+c;\ D_3=-a+3b-3c+d;$$
$$D_4=a-4b+6c-4d+e, \text{ etc.}$$

Here, the coëfficients of a, b, c, d, etc., in the n^{th} order of differences, are evidently those of the terms of a binomial raised to the n^{th} power; and their signs are alternately positive and negative. Hence, the first term of the n^{th} order of differences is

$$a-nb+\frac{n(n-1)}{1\cdot 2}c-\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}d+, \text{ etc., when } n \text{ is } \textit{even}, \text{ and}$$

$$-a+nb-\frac{n(n-1)}{1\cdot 2}c+\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}d-, \text{ etc., when } n \text{ is } \textit{odd}.$$

Corollary.—It is evident from the coëfficients that when $n=1$, the value of D_n has only *two* terms, for then $n-1=0$; when $n=2$, this value has only *three* terms, for then $n-2=0$, and so on.

1. Find the first term of the fourth order of differences of the series $1^3, 2^3, 3^3, 4^3, 5^3, \ldots$ or 1, 8, 27, 64, etc. . . .

Here, $n=4$; hence, take five terms of the first value of D_n, and $a=1$, $b=8$, $c=27$, $d=64$, $e=125$, and $D_4=$

$$1-4\times8+\frac{4\times3}{1\times2}\times27-\frac{4\times3\times2}{1\times2\times3}\times64+\frac{4\times3\times2\times1}{1\times2\times3\times4}\times125=$$
$$1-32+162-256+125=0, \text{ Ans.}$$

2. Find the first term of the second order of differences of the series $1^2, 2^2, 3^2, 4^2, \ldots$ or 1, 4, 9, 16, 25.
Ans. 2.

3. What is the first term of the third order of differences of the series 1, 3, 6, 10, 15, etc.? Ans. 0.

4. Required the first term of the fifth order of differences of the series $1, 3, 3^2, 3^3, 3^4$, etc. Ans. 32.

5. Find the first term of the fifth order of differences of the series $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}$, etc. Ans. $-\frac{1}{32}$.

326. Problem II.—*To find the n^{th} term of the series* a, b, c, d, e, etc.

From the preceding article, we have seen that

$D_1=-a+b$; whence $b=a+D_1$;
$D_2=a-2b+c$; " $c=a+2D_1+D_2$;
$D_3=-a+3b-3c+d$; " $d=a+3D_1+3D_2+D_3$;
$D_4=a-4b+6c-4d+e$; " $e=a+4D_1+6D_2+4D_3+D_4$.
. . . .

It is evident from inspection that the coëfficients of the n^{th} term of the series are the coëfficients of the $(n-1)$ power of a binomial.

Hence, writing $n-1$ instead of n, in the coëfficients of the n^{th} power of $a+b$, (Art. 319,) the n^{th} term of the series is

$$a+(n-1)D_1+\frac{(n-1)(n-2)}{1\cdot 2}D_2+\frac{(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3}D_3+\text{ etc.}$$

1. Find the 12^{th} term of the series 1, 3, 6, 10, 15, 21, . .

1 , 3 , 6 , 10 , 15, ...		
2 , 3 , 4 , 5, ...	hence,	$D_1=2$;
1 , 1 , 1, ...	"	$D_2=1$;
0 , 0, ...	"	$D_3=0$;

Or, D_1, D_2, D_3, etc., may be found from the formula, (Art. 325,) and the succeeding orders of differences are also evidently 0; hence, 12^{th} term

$$a+(n-1)D_1+\frac{(n-1)(n-2)}{1\cdot 2}D_2=1+11\times 2+\frac{11\cdot 10}{2}\times 1$$

$$=1+22+55=78, \text{ Ans.}$$

2. Find the n^{th} term of the series 2, 6, 12, 20, 30, . .

Proceeding as above, to find the orders of differences, we have

$$D_1=4,\quad D_2=2, \text{ and } D_3=0;$$

hence, n^{th} term $=2+(n-1)4+\frac{(n-1)(n-2)}{1\cdot 2}\times 2=n^2+n$, Ans.

From the formula n^2+n, or $n(n+1)$, any term of this series is readily found; thus, the 20^{th} term $=20(20+1)=420$.

It is also evident that the n^{th} term of a series can be found exactly, only when some order of differences is zero.

3. Find the 15^{th} term and the n^{th} term of the series 1, 2^2, 3^2, 4^2, . . or 1, 4, 9, 16, . . Ans. 225, and n^2.

4. Find the 12^{th} term of the series 1, 5, 15, 35, 70, 126, etc. Ans. 1365.

5. Find the n^{th} term of the series 1, 3, 6, 10, etc.

Ans. $\frac{n(n+1)}{2}$.

6. Find the 9^{th} term of the series $2 \cdot 5 \cdot 7$, $4 \; 7 \cdot 9$, $6 \cdot 9 \cdot 11$, $8 \cdot 11 \; 13$, etc. Ans. 8694.

7. What is the n^{th} term of the series 1×2, 3×4, 5×6, etc.? Ans. $4n^2-2n$.

327. Problem III.—*To find the sum of n terms of the series* a, b, c, d, c, etc.

Assume the series $0, a, a+b, a+b+c, a+b+c+d, \ldots$

Subtracting each term from the next succeeding, we have

$$a, \; b, \; c, \; d, \; e, \text{ etc.},$$

which is the series whose sum it is proposed to find. Hence, the sum of n terms of the proposed series, which it is now required to find, is the $(n+1)^{th}$ term of the assumed series.

It is evident the n^{th} order of differences in the given series is equal to the $(n+1)^{th}$ order in the assumed series. Hence, if we compare the quantities in the assumed series, with those of the formula for finding the n^{th} term of a series (Art. 326), we have

0 for a,	a for D_1,
$n+1$ for n,	D_1 for D_2, etc.

Substituting these values in the formula, we have $0+(n+1-1)a$

$$+\frac{(n+1-1)(n+1-2)}{1 \cdot 2}D_1+\frac{(n+1-1)(n+1-2)(n+1-3)}{1 \cdot 2 \cdot 3}D_2+, \ldots$$

Or, $\quad . \; na+\frac{n(n-1)}{1 \cdot 2}D_1+\frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3}D_2+$, etc., which is the sum of n terms of the proposed series.

1. Find the sum of n terms of the odd numbers 1, 3, 5, 7, 9, $\ldots$

Here, $a=1$, $D_1=2$, $D_2=0$; hence,

$$\text{Sum } =na+\frac{n(n-1)}{1 \cdot 2}D_1=n\times 1+\frac{n(n-1)}{2}\times 2=n+n^2-n=n^2.$$

2. Find the sum of n terms of the series 1^2, 2^2, 3^2, 4^2, 5^2, $\ldots$

Here, $a=1$, $D_1=3$, $D_2=2$, $D_3=0$; hence,

$$\text{Sum} = na+\frac{n(n-1)}{1\cdot 2}D_1+\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}D_2=n+\frac{3n(n-1)}{2}$$

$$+\frac{n(n-1)(n-2)}{3}=\frac{n(n+1)(2n+1)}{6}.$$

3. Find the sum of n terms of the series $1+3+6+10+15$, etc. Ans. $\frac{n(n+1)(n+2)}{6}$.

4. Find the sum of 20 terms of the series $3+11+31+69+131$, etc. Ans. 44330.

5. Find the sum of 20 terms of the series
$1\cdot 2\cdot 3+2\quad 3\cdot 4+3\cdot 4\cdot 5+$, etc. Ans. 53130.

6. Find the sum of n terms of the series of cube numbers $1^3+2^3+3^3+$, etc. Ans. $[\frac{1}{2}n(n+1)]^2$.

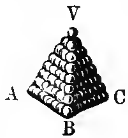
7. Find the sum of 25 terms of the series whose n^{th} term is $n^2(3n-2)$. Ans. 305825.

PILING OF CANNON BALLS AND SHELLS.

328. Balls and shells are usually piled by horizontal courses, either in the form of a pyramid or a wedge; the base being either an equilateral triangle, or a square, or a rectangle. In the triangle and square, the pile terminates in a single ball, but in the rectangle it finishes in a ridge, or single row of balls.

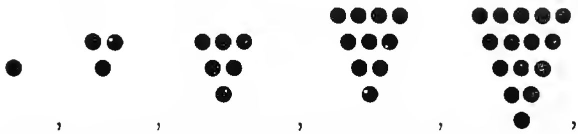
329. *To find the number of balls in a triangular pile.*

A triangular pile, as V—ABC, is formed of successive horizontal courses of the form of an equilateral triangle, the number on each side decreasing continually by unity from the bottom to the single ball at the top.



If we commence at the top, the number of balls in the respective courses will be as follows:

1st. 2d. 3d. 4th. 5th.



and so on. Hence, the number of balls in the respective courses is 1, $1+2$, $1+2+3$, $1+2+3+4$, $1+2+3+4+5$, and so on; or,
1 3 6 10 15

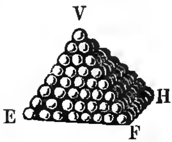
Hence, to find the number of balls in a triangular pile, is to find the sum of the series 1, 3, 6, 10, 15, etc., to as many terms (n) as there are balls in one side of the lowest course.

By applying the formula (Art. 327) to finding the sum of n terms of the series 1, 3, 6, 10, etc., we have $a=1$, $D_1=2$, $D_2=1$, and $D_3=0$.

Hence, the formula $na+\frac{n(n-1)}{1\cdot 2}D_1+\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}D_2$ gives

$$n+n^2-n+\frac{n^3-3n^2+2n}{6}=\frac{n(n+1)(n+2)}{6}. \qquad (A)$$

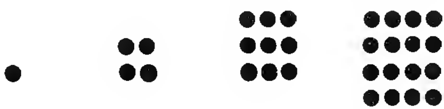
330. *To find the number of balls in a square pile.*



A square pile, as V—EFH, is formed of successive square horizontal courses, such that the number of balls in the sides of these courses decreases continually by unity, from the bottom to the single ball at the top.

If we commence at the top, the number of balls in the respective courses will be as follows:

1st. 2d. 3d. 4th.

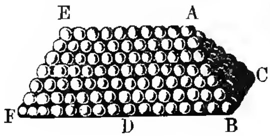


and so on. Hence, the number of balls in the respective courses is 1^2, 2^2, 3^2, 4^2, 5^2, etc., or 1, 4, 9, 16, 25, and so on. Therefore, to find the number of balls in a square pile, is to find the sum of the squares of 1, 2, 3, etc., to as many terms (n) as there are balls in one side of the lowest course.

But this sum (Ex. 2, pp. 297, 298) is $\frac{n(n+1)(2n+1)}{6}$. (B)

331. *To find the number of balls in a rectangular pile.*

A rectangular pile, as EFD BCA, is formed of successive rectangular courses, the number of balls in each of the sides decreasing by unity from the bottom to the single row at the top.



If we commence at the top, the number of balls in the *breadth* of the successive rows is 1, 2, 3, and so on. Also, if $m+1$ denotes the number of balls in the top row, the number in the *length* of the second row will be $m+2$, in the third, $m+3$, and so on. Hence, the number in the respective courses, commencing with the top, will be $1(m+1)$, $2(m+2)$, $3(m+3)$, and in the n^{th} course $n(m+n)$. Or,

$$S=1(m+1)+2(m+2)+3(m+3)+\ .\ .\qquad +n(m+n)$$
$$=m(1+2+3+4\qquad .\ +n)+(1^2+2^2+3^2+4^2+\qquad +n^2);$$

but the sum of n terms of the series in the two parentheses (Arts. 327, 330,) is $\frac{n(n+1)}{2}$, and $\frac{n(n+1)(2n+1)}{6}$. Hence,

$$S=\frac{mn(n+1)}{2}+\frac{n(n+1)(2n+1)}{6}=\frac{n(n+1)}{6}(3m+2n+1)\qquad (C).$$

Here, $m+n$ represents the number of balls in the length of the lowest course. If we put $m+n=l$, we have $3m+2n=3l-n$; substituting this for $3m+2n$, in (C), we have

$$S=\frac{n(n+1)}{6}(3l-n+1).$$

It is evident that the number of courses in a triangular or square pile is equal to the number of balls in one side of the base course, and in the rectangular pile to the number of balls in the *breadth* of the base course.

332. Collecting together the results of the three preceding articles, we have for the number of balls in a

Triangular pile $\frac{1}{6}n(n+1)(n+2)$ . . . . (A);

Square pile $\frac{1}{6}n(n+1)(2n+1)$ . . . . (B);

Rectangular pile $\frac{1}{6}n(n+1)(3l-n+1)$ . . . (C).

In (A) and (B), n denotes the number of courses, or number of balls in the base course. In (C), n denotes the number in the breadth, and l the number in the length, of the base course.

The number of balls in an *incomplete* pile is evidently found by subtracting the number in the pile which is wanting at the top, from the whole pile considered as complete.

1. Find the number of balls in a triangular pile of 15 courses. Ans. 680.

Here, $n=15$. Substituting this value in (A), we find the number

$$=\frac{15(15+1)(15+2)}{2\times3}=\frac{15\times16\times17}{6}=680, \text{ Ans.}$$

2. Find the number of balls in an incomplete triangular pile of 15 courses, having 21 balls in the upper course.

From the illustrations in Art. 329, it is evident that the number of balls in one side of the upper course is 6; therefore, 5 courses have been removed from the pile. From formula (A), we find that the pile, considered as complete, would contain 1540 balls, and that the removed pile contains 35. Hence, 1540—35=1505, the number left.

3. Find the number of balls in a square pile of 15 courses. Ans. 1240.

4. Find the number of balls in a rectangular pile, the length and breadth of the base containing 52 and 34 balls respectively. Ans. 24395.

5. Find the number of balls in an incomplete triangular pile, a side of the base course having 25 balls, and a side of the top 13. Ans. 2561.

6. How many balls in an incomplete triangular pile of 15 courses, having 38 balls in a side of the base?

Ans. 7580.

7. Find the number of balls in an incomplete square pile, a side of the base course having 44 balls, and a side of the top 22. Ans. 26059.

8. The whole number of balls in the base and top courses of a square pile are 1521 and 169 respectively; how many are in the incomplete pile? Ans. 19890.

9. The number of balls in a complete rectangular pile of 20 courses is 6440; how many balls are in its base?

Ans. 740.

10 The number of balls in a triangular pile is to the number in a square pile having the same number of balls in the side of the base as 6 to 11; required the number in each pile. Ans. 816, and 1496.

11. How many balls are in an incomplete rectangular pile of 8 courses, having 36 balls in the longer side, and 17 in the shorter side of the upper course? Ans. 6520.

INTERPOLATION OF SERIES.

333. Interpolation is the process of finding intermediate numbers in mathematical, astronomical, or other tables. Its object is to furnish a shorter method of completing such tables when portions of them have been calculated by formulæ.

Thus, if the logarithms of 5, 6, and 8, are respectively 0.6989, 0.7782, and 0.9031, it may be required from these data to find the logarithm of 7.

The latter numbers are sometimes called *functions* of the former, and the former *arguments* of the functions.

As the functions constitute a series, the principle upon which interpolation is founded is explained in Art. 326; that is, certain terms of a series being known, it is required to find the n^{th} term.

Three cases may arise, which we will now consider.

Case I.—When the differences of the *functions* are proportional, or nearly proportional, to the differences of the *arguments*, or the functions are in arithmetical progression.

Ex.—Given the Dip of the Sea Horizon at the heights of 86, 89, 92, 95, and 98 feet, viz., 9′08″, 9′17″, 9′26″, 9′36″, and 9′45″; required that of 101 feet.

Ans. 9′54″.

Here, the first differences being 9″, or nearly so, we add 9″ to 9′45″ for the Dip at 101 feet.

In all practical examples, there is no common first difference, and it becomes necessary to employ the second, third, etc., differences. If in the series composing the functions, we can obtain an order of differences equal to zero, the interpolation will be exact. In most cases, however, D_2, D_3, etc., do not vanish, but become so small after D_2 or D_3 that they may be omitted without sensible error.

334. Case II.—When the differences of the functions are *not* proportional to the differences of the arguments, and *the term to be interpolated is one of the equidistant functions.*

Ex.—Given $\sqrt[3]{25}=2.92401$, $\sqrt[3]{26}=2.96249$, $\sqrt[3]{27}=3$, $\sqrt[3]{29}=3.07231$, to find the cube root of 28.

In such examples, when three quantities are given, we may suppose D_3 to vanish or become very small. We then have (Art. 326) the equation $-a+3b-3c+d=0$, and any of the quantities a, b, c, or d, may be found, when the other three are given. Similarly, if the fourth differences vanish, then

$$a-4b+6c-4d+e=0.$$

In the above example, four quantities are given to find a fifth; therefore, we have $a-4b+6c-4d+e=0$, where d is the term to be interpolated; hence, $4d=a+6c+e-4b=2.92401+18+3.07231-11.84996=12.14636$, where d, or $\sqrt[3]{28}=3.03659$, which is true to .00001.

335. Case III.—When the differences are as in Case 2d, and the term to be interpolated is *intermediate* to any two of the functions.

Ex.—Having given the logarithms of 102, 103, 104, and 105, let it be required to find the logarithm of 103.55.

Taking the formula, Art. 326, put p to represent the distance, *in intervals*, of the required term (t) from a, the first term of the series, in which case $p=n-1$, since the number of *intervals* is one less than the number of terms. Then,

$$t=a+pD_1+\frac{p(p-1)}{1\cdot 2}D_2+\frac{p(p-1)(p-2)}{1\cdot 2\cdot 3}D_3+, \text{ etc.}$$

The intervals between the given numbers is always to be considered as *unity*, and p is to be reckoned in parts of this interval; hence, p will be fractional.

Sufficient accuracy is generally obtained by making use of D_1 and D_2 only, in the above formula.

In practice, however, the following is generally adopted:

Take the two *functions* of the series which precede, and the two which follow the term required, and find from them the three first differences, and the two second differences. Call the *second* of the three first differences d, the *mean* of the two second differences d', the fractional part of the interval p', and second term b. We then have from the above formula,

$$t=b+p'(d+\frac{p'-1}{2}d').$$

Applying this formula to the above example, we have

Nos.	Logarithms.	1st Diff.	2d Diff.	Mean of 2d Diff.
102	2.0086002			
103	2.0128372	42370	—409	
104	2.0170333	41961	—401	—405
105	2.0211893	41560		

Here, $p'=.55$, $d=41961$, $d'=-405$, and $b=2.0128372$.

$t=2.0128372+.55(41961+\frac{.45}{2}\times 405)$.

$t=2.0128372+.0023129=2.0151501$, Ans.

1. Find the 2^d term of the series of which the 4^{th} differences vanish, the 1^{st}, 3^d, 4^{th}, and 5^{th} terms being 3, 15, 30, 55; and find the 6^{th}, 7^{th}, and 8^{th} terms.

Ans. 7; and 93, 147, and 220.

2. Find the 5^{th} term of the series of which the 6^{th} differences vanish, and the 1^{st}, 2^d, 3^d, 4^{th}, 6^{th}, and 7^{th} terms are 11, 18, 30, 50, 132, 209. Ans. 82

3. Given the logarithms of 101, 102, 104, and 105; viz., 2.0043214, 2.0086002, 2.0170333, and 2.0211893, to find the logarithm of 103. Ans. 2.0128372.

4. Given the cube roots of 60, 62, 64, and 66; viz., 3.91487, 3.95789, 4, and 4.04124, to find the cube root of 63. Ans. 3.97905.

5. Having given the squares of any two consecutive whole numbers, show how the squares of the succeeding whole numbers may be obtained by addition.

INFINITE SERIES.

336. An **Infinite Series** is a series consisting of an unlimited number of terms.

The **Sum** of an infinite series is the *limit* to which we approach by adding together more terms, but which can

not be exceeded by adding together any number of terms whatever.

A **Convergent Series** is one which has a *sum* or *limit*.

Thus, $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\frac{1}{64}+$, etc.,

is a convergent series, whose limit is 2, since the sum of any number of its terms can not exceed 2.

A **Divergent Series** is one which has no *sum* or *limit;* as,

$$1+2+4+8+16+32+, \text{ etc.}$$

An **Ascending Series** is one in which the powers of the leading quantity continually increase; as,

$$a+bx+cx^2+dx^3+.$$

A **Descending Series** is one in which the powers of the leading quantity continually diminish; as,

$$a+bx^{-1}+cx^{-2}+dx^{-3}+, \text{ or } a+\frac{b}{x}+\frac{c}{x^2}+\frac{d}{x^3}+.$$

337. There are *four* general methods of converting an algebraic expression into an infinite series of equivalent value, each of which has been already exemplified; viz.,

1st. By *Division*, Art. 134; 2d. By *Extraction of Roots*, Art. 183; 3d. By *Indeterminate Coëfficients*, Arts. 315–7; and, 4th. By the *Binomial Theorem*, Art. 321.

338. The **Summation of a Series** is the finding a finite expression equivalent to the series.

The **General Term of a Series** is an expression from which the several terms of the series may be derived according to some determinate law.

Thus, in the series $\frac{a}{1}+\frac{a}{2}+\frac{a}{3}+\frac{a}{4}+\ldots\ldots$ the general term is $\frac{a}{x}$, because by making $x=1$, 2, 3, etc., each term of the series is found.

Again, in the series $2\cdot2+2\cdot3+2\cdot4+2\cdot5+\ldots\ldots$ the general term is $2(x+1)$.

As different series are in general governed by different laws, the methods of finding the sum, which are applicable to one class, will not apply universally.

We present two methods of most general application.

FIRST METHOD.—In a regular decreasing geometrical series, whose first term is a, and ratio r, the sum is $\frac{a}{1-r}$ (Art. 299).

SECOND METHOD.—By subtraction.

Ex. 1.—Find the sum of the infinite series $\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\frac{1}{4\cdot 5}+\frac{1}{5\cdot 6}+$, etc.

Put $\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+$, etc., $=S$;

Then, $\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+$, etc., $=S-\frac{1}{2}$.

Subtracting $\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\frac{1}{4\cdot 5}+\frac{1}{5\cdot 6}+$, etc., $=\frac{1}{2}$, Ans.

Ex. 2.—Find the sum of the infinite series $\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\frac{1}{5\cdot 7}+\frac{1}{7\cdot 9}+$, etc.

Put $\frac{1}{1}+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+$, etc., $=S$;

Then, $\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{9}+$ etc., $=S-1$.

Subtracting $\frac{2}{1\cdot 3}+\frac{2}{3\cdot 5}+\frac{2}{5\cdot 7}+$, etc, $=1$, and $\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\frac{1}{5\cdot 7}+$, etc., $=\frac{1}{2}$, Ans.

In such series, the first factor in the successive denominators is variable, while the second factor exceeds the first by a constant quantity. The general term is therefore $\frac{q}{n(n+p)}$, where n is variable and p constant.

Since $\frac{q}{n}-\frac{q}{n+p}=\frac{pq}{n(n+p)}$ $\therefore$ $\frac{q}{n(n+p)}=\frac{1}{p}\left\{\frac{q}{n}-\frac{q}{n+p}\right\}$.

From which we derive the following

Rule.—*Having found the values of q, n, and p, in the given series, express the series whose general formulas are $\frac{q}{n}$ and $\frac{q}{n+p}$; subtract the latter from the former, and divide the result by p for the sum of the series.*

1. Required the sum of the series $\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\frac{1}{5\cdot 7}+$, etc., ad infinitum, that is, *to infinity.*

Here, . . $q=1$, $p=2$, and $n=1$, 3, 5, 7, etc.

Put . $\frac{q}{n(n+p)}=\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\frac{1}{5\cdot 7}+$, etc.

Then, $\frac{q}{n}=\frac{1}{1}+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+$, etc., ad inf.

And . $\frac{q}{n+p}=\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+$, etc., ad inf.

Subtracting, $\frac{q}{n}-\frac{q}{n+p}=\frac{pq}{n(n+p)}=1$;

$\frac{q}{n(n+p)}=\frac{1}{2}=$ sum of given series.

The sum of n terms of the same series is found in a manner nearly similar. Thus,

$$\frac{q}{n}=1+\tfrac{1}{3}+\tfrac{1}{5}+\tfrac{1}{7}\ \cdots\cdots\ \frac{1}{2n-1}$$

$$\frac{q}{n+p}=\tfrac{1}{3}+\tfrac{1}{5}+\tfrac{1}{7}\cdots\cdots\ \frac{1}{2n-1}+\frac{1}{2n+1}$$

$\frac{q}{n}-\frac{q}{n+p}=\frac{pq}{n(n+p)}=1-\frac{1}{2n+1}=\frac{2n}{2n+1}$ and $\frac{q}{n(n+p)}=\frac{n}{2n+1}$, Ans.

2. Find the sum of the series $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+$, etc., ad infinitum.

Here, $q=1$, $p=1$, and $n=1$, 2, 3, etc. Ans. 1.

3. Find the sum of the above series to n terms.

Ans. $\frac{n}{n+1}$.

4. Find the sum of the series $\frac{1}{1\cdot 4}+\frac{1}{2\cdot 5}+\frac{1}{3\cdot 6}+\frac{1}{4\cdot 7}$ $+$, etc., ad infinitum.

Here, $q=1$, and $p=3$, $n=1$, 2, 3, etc. Ans. $\frac{11}{18}$.

5. Find the sum of the series $\frac{1}{1\cdot 3}+\frac{1}{2\cdot 4}+\frac{1}{3\cdot 5}+$, etc., ad infinitum.

Here, $q=1$, $p=2$, and $n=1$, 2, 3, etc. Ans. $\frac{3}{4}$.

6. Find the series whose general term is $\frac{1}{n(n+4)}$; also find its sum continued to infinity.

A. Series $=\frac{1}{1\cdot 5}+\frac{1}{2\cdot 6}+\frac{1}{3\cdot 7}+\frac{1}{4\cdot 8}+$, etc., sum $=\frac{25}{48}$.

The sums of series may often be found by reducing them, by multiplication or division, to the forms already known. Thus,

7. Find the sum of the series $1+\frac{1}{3}+\frac{1}{6}+\frac{1}{10}+$, etc., ad infinitum. Ans. 2.

Divide by 2 and compare with example 2d.

8. Find the sum of the series $\frac{1}{3\cdot 8}+\frac{1}{6\cdot 12}+\frac{1}{9\cdot 16}+$, etc., ad infinitum. (Multiply by $3\cdot 4$). Ans. $\frac{1}{12}$.

Remark.—There are other methods for the summation of certain classes of series, but they are too complex and extensive for an elementary work.

RECURRING SERIES.

339. A **Recurring Series** is a series so constituted that every term is connected with one or more of the terms which precede it by an invariable law, usually dependent on the operations of addition, subtraction, etc.

Thus, in the series $1+2x+3x^2+5x^3+8x^4+13x^5+21x^6+$, etc., the sum of the coëfficients of any two consecutive terms is equal

to the coëfficient of the next following term; and, by means of this relation between the coëfficients, the series may be extended to any desired number of terms.

340. The particular relation, by means of which the coëfficient of any term of the series may be found when the preceding coëfficients are known, is called the *scale of the coëfficients.* It is easily seen that it is sufficient to find the successive *coëfficients* in order to determine the series, inasmuch as the desired powers of the variable may be supplied as wanted.

Recurring series are of the *first order*, *second order*, etc., according to the number of terms in the scale.

Thus, in the series $1-\frac{b}{a}x+\frac{b^2}{a^2}x^2-\frac{b^3}{a^3}x^3$, etc., the coëfficient of each term after the first is equal to the preceding coëfficient multiplied by $-\frac{b}{a}$, and the series is said to be of the first order. This, the simplest form of a recurring series, is obviously a series in Geometrical Progression.

341. *To find the scale of the coëfficients of a recurring series.*

When the series is of the first order, the scale is easily determined, being the ratio of any two consecutive coëfficients. (Art. 295.)

When the series is of the *second order*, the law of the series depends on two terms, and the scale consists of two parts.

Let p and q represent the two terms of the scale of the coëfficients of the recurring series,

$$A+Bx+Cx^2+Dx^3+Ex^4+Fx^5, \text{ etc.},$$

Then, by the assumed law of the series:

$$\begin{aligned} C&=Bp+Aq; && (1)\\ D&=Cp+Bq; && (2)\\ E&=Dp+Cq;\ \text{etc.} && (3)\end{aligned}$$

The values of p and q may be found by eliminating between any two of these equations. Taking the first two, (Art. 158.)

$$\therefore\ p=\frac{BC-AD}{B^2-AC}\ \text{ and } q=\frac{BD-C^2}{B^2-AC}$$

Ex.—Find the scale of the series $1+2x+3x^2+4x^3+5x^4$, etc.

Here, $A=1$, $B=2$, $C=3$, $D=4$, etc.

$$\therefore\ p=\frac{2\times3-1\times4}{2^2-1\times3}=2\ \text{ and } q=\frac{2\times4-3^2}{2^2-1\times3}=-1.$$

Now, by the use of the scale, we may extend the series as far as we please: the 5th coëfficient $=p\times$ the 4th$+q\times$ the 3d$=2\times4-3=5$; the 6th coëfficient$=2\times5-4=6$; the 7th$=2\times6-5=7$, and as the ascending powers of x are wanted, we have $6x^5$ for the 6th term, $7x^6$ for the 7th, etc.

342. In a recurring series of the *third order*, the law of the series involves three terms, which we will represent by p, q, and r, the series being $A+Bx+Cx^2+Dx^3+Ex^4+Fx^5+Gx^6$, etc.

Then, by the law of the series,

$$\begin{aligned} D&=Cp+Bq+Ar;\\ E&=Dp+Cq+Br;\\ F&=Ep+Dq+Cr;\ \text{etc.,}\end{aligned}$$

And, by combining these equations, the values of p, q, and r are readily found, (Art. 158.) In a similar manner the scale may be determined in series of the higher orders.

In finding the scale of a series, we must first ascertain by inspection whether the series is in G. P.; if not, then

make trial of a scale containing two terms, then one of three, four, and so on, until a correct result is obtained. We must be careful *not to assume too many terms*; for in that case every term of the scale will take the form $\frac{0}{0}$.

343. *To find the sum of an infinite recurring series whose scale of relation is known.*

Let $A+Bx+Cx^2+Dx^3+Ex^4$, etc., be a recurring series of the second order, p and q being the terms of the scale.

Then, . . . $A=A$;
$Bx=Bx$;
$Cx^2=Bpx^2+Aqx^2$;
$Dx^3=Cpx^3+Bqx^3$; etc., *ad infinitum.*

Represent by S the required sum, and add together the corresponding members of the preceding equations, observing that $Bx+Cx^2+Dx^3+$, etc., $=S-A$; then, we have

$$S=A+Bx+(S-A)px+Sqx^2;$$

$$\therefore\ S-Spx-Sqx^2=A+Bx-Apx;$$

Or, ... $S=\dfrac{A+Bx-Apx}{1-px-qx^2}$.

If we make $q=0$, (remembering that $B=Ap$), the formula becomes $S=\dfrac{A}{1-px}$, which is, as it ought to be, identical with the formula of Art 299.

REMARK.—Every recurring series may be supposed to arise from the development of a rational fraction, and the summation of such a series may be regarded as a return to the generating fraction. There are several methods of accomplishing this return: of these the preceding is regarded as the most suitable for an elementary work.

1. Find the sum of $1+3x+5x^2+7x^3+9x^4$, etc.

Here, $A=1$, $B=3$, $C=5$, $D=7$, etc.

And, hence, (Art. 341.) $p=2$, $q=-1$.

Then, $S=\dfrac{A+Bx-Apx}{1-px-qx^2}=\dfrac{1+3x-2x}{1-2x+x^2}=\dfrac{1+x}{(1-x)^2}$.

In each of the following series, find the scale of relation, and the sum (S) of an infinite number of terms:

2. $1+6x+12x^2+48x^3+120x^4+$, etc.

Ans. $p=1,\ q=6$; $S=\dfrac{1+5x}{1-x-6x^2}$.

3. $1+2x+3x^2+4x^3+5x^4+6x^5+$, etc.

Ans. $p=2,\ q=-1$; $S=\dfrac{1}{(1-x)^2}$.

4. $\dfrac{a}{c}-\dfrac{abx}{c^2}+\dfrac{ab^2x^2}{c^3}-\dfrac{ab^3x^3}{c^4}+$, etc.

Ans. The series is in G. P. $p=-\dfrac{b}{c}$; $S=\dfrac{a}{c+bx}$.

5. $x+x^2+x^3+$, etc.

Ans. The series is in G. P. $p=1$; $S=\dfrac{x}{1-x}$.

6. $x-x^2+x^3-x^4+$, etc.

Ans. The series is in G. P. $p=-1$; $S=\dfrac{x}{1+x}$.

7. $1+3x+5x^2+7x^3+9x^4+$, etc.

Ans. $p=2,\ q=-1$, $S=\dfrac{1+x}{1-2x+x^2}$.

8. $1^2+2^2x+3^2x^2+4^2x^3+5^2x^4+6^2x^5+$, etc.

Ans. $p=3,\ q=-3,\ r=1$; $S=\dfrac{1+x}{(1-x)^3}$.

REVERSION OF SERIES.

344. To Revert a Series is to express the value of the unknown quantity in it by means of another series involving the powers of some other quantity.

Let x and y represent two undetermined quantities, and express the value of y by a series involving the powers of x; thus,

$$y=ax+bx^2+cx^3+dx^4+\text{, etc.,} \qquad (1),$$

in which a, b, c, d, etc., are known quantities; then, to *revert* this series is to express the value of x in a series

containing the known quantities a, b, c, d, etc., and the powers of y.

To revert this series, assume $x=Ay+By^2+Cy^3+Dy^4$, etc. (2), in which the coëfficients A, B, C . . . are undetermined.

Find the values of y^2, y^3, y^4 . . . from (1), thus,

$$\begin{aligned} y^2 &= a^2x^2+2abx^3+(b^2+2ac)x^4+\ \ldots \\ y^3 &= \qquad\qquad a^3x^3+3a^2bx^4+\ \ldots \\ y^4 &= \qquad\qquad\qquad\qquad a^4x^4+\ \ldots \text{ etc.} \end{aligned}$$

Substituting these values in (2), and arranging, we have

$$\begin{array}{r|r|r|r|l} 0=Aa & x+Ab & x^2+\quad Ac & x^3+\quad Ad & x^4+\text{, etc.} \\ -1 & Ba^2 & +2Bab & +\quad Bb^2 & \\ & & +\ Ca^3 & +\ 2Bac & \\ & & & +3Ca^2b & \\ & & & +\quad Da^4 & \end{array}$$

Since this is true, whatever be the value of x, and the coëfficients of x, x^2, x^3, etc., will each $=0$, (Art. 314, Cor.), we have

$$Aa-1 \ \ldots\ldots\ldots =0, \therefore A=\frac{1}{a},$$

$$Ab+Ba^2 \ \ldots\ldots\ldots =0, \therefore B=-\frac{b}{a^3},$$

$$Ac+2Bab+Ca^3 \ \ldots\ldots =0, \therefore C=\frac{2b^2-ac}{a^5},$$

$$Ad+Bd^2+2Bac+3Ca^2b+Da^4=0, \therefore D=-\frac{a^2d-5abc+5b^3}{a^7}$$

Hence, $x=\frac{1}{a}y-\frac{b}{a^3}y^2+\frac{2b^2-ac}{a^5}y^3-\frac{a^2d-5abc+5b^3}{a^7}y^4+$, etc. (3)

345. If the given series has a constant term prefixed, thus,
$$y=a'+ax+bx^2+cx^3+dx^4+\ \ldots\ldots$$
assume $y-a'=z$, and we have
$$z=ax+bx^2+cx^3+dx^4+\text{, etc.}$$

But this is the same as (1) in the preceding article, except that z stands in the place of y; hence, if z be substituted for y in

(3,) [Art. 344], the result will be the required development of x; and then, $y-a'$ being substituted for z, the result is

$$x=\frac{1}{a}(y-a')-\frac{b}{a^3}(y-a')^2+\frac{2b^2-ac}{a^5}(y-a')^3-\text{, etc.}$$

346. When the given series contains the odd powers of x, assume for x another series containing the odd powers of y. Thus, if $y=ax+bx^3+cx^5+dx^7+\ldots\ldots$

to develope x in terms of y, assume

$$x=Ay+By^3+Cy^5+Dy^7+\ldots\ldots$$

Then, by substituting the values of y, y^2, etc., derived from the former equation, in the latter, and equating the coëfficients to zero, we find

$$x=\frac{1}{a}y-\frac{b}{a^4}y^3+\frac{3b^2-ac}{a^7}y^5-\frac{a^2d-8abc+12b^3}{a^{10}}y^7+\text{, etc.}$$

If both sides of the equation be expressed in a series, as

$$ay+by^2+cy^3+\text{, etc., }=a'x+b'x^2+c'x^3+\text{, etc.,}$$

and it be required to find y in terms of x, we must assume, as before,

$$y=Ax+Bx^2+Cx^3+Dx^4+\text{, etc.,}$$

and substitute the values of y, y^2, y^3, etc., derived from this last equation, in the proposed equation; we shall then, by equating the coëfficients of the like powers of x, determine the values of A, B, C, etc., as before.

The following exercises may be solved either by substituting the values of a, b, c, etc., in the equations obtained in the preceding articles, or by proceeding according to the methods by which those equations were obtained.

1. Given the series $y=x-x^2+x^3-x^4+\ldots$ to find the value of x in terms of y. Ans. $x=y+y^2+y^3+y^4+$, etc.

Find the value of x, in an infinite series in terms of y:

2. When $y=x+x^2+x^3+$, etc.

Ans. $x=y-y^2+y^3-y^4+y^5-$, etc.

3. When $y=2x+3x^3+4x^5+5x^7+$, etc.

Ans. $x=\frac{1}{2}y-\frac{3}{16}y^3+\frac{19}{128}y^5-$, etc.

4. When $y=1-2x+3x^2$.

Ans. $x=-\frac{1}{2}(y-1)+\frac{3}{8}(y-1)^2-\frac{9}{16}(y-1)^3+$, etc.

5. When $y=x+\frac{1}{2}x^2+\frac{1}{6}x^3+\frac{1}{24}x^4+$, etc.

Ans. $x=y-\frac{1}{2}y^2+\frac{1}{3}y^3-\frac{1}{4}y^4+$, etc.

6. When $y+ay^2+by^3+cy^4 \ldots =gx+hx^2+kx^3+lx^4$.

Ans. $x=\dfrac{y}{g}+\dfrac{(ag^2-h)y^2}{g^3}+\dfrac{[bg^4-kg-2h(ag^2-h)]y^3}{g^5}+\ldots$

XI. CONTINUED FRACTIONS: LOGARITHMS: EXPONENTIAL EQUATIONS: INTEREST, AND ANNUITIES.

CONTINUED FRACTIONS.

347. **A Continued Fraction** is one whose denominator is continued by being itself a *mixed number*, and the denominator of the fractional part again continued as before, and so on; thus,

$$\cfrac{1}{a+\cfrac{1}{b}},\quad \cfrac{1}{a+\cfrac{1}{b+\cfrac{1}{c}}},\quad \cfrac{1}{a+\cfrac{1}{b+\cfrac{1}{c+\cfrac{1}{d}}}}$$

in which a, b, c, d, etc., are positive whole numbers.

Continued fractions are useful in *approximating* to the values of ratios expressed by *large numbers*, in resolving exponential equations, indeterminate equations of the first degree, etc.

348. *To express a rational fraction in the form of a continued fraction.*

Let it be required to reduce $\frac{30}{157}$ to a continued fraction.

If we divide both terms of the fraction by the numerator, we find $\frac{30}{157}=\cfrac{1}{5+\frac{7}{30}}$.

If we omit $\frac{7}{30}$, the denominator will be too *small*, and $\frac{1}{5}$, the value of the fraction, will be too *large*.

Again, if we divide both terms of the fraction $\frac{7}{30}$ by the numerator, we find $\frac{30}{157}=\cfrac{1}{5+\cfrac{1}{4+\frac{2}{7}}}$.

If we omit $\frac{2}{7}$, the value will be expressed by $\cfrac{1}{5+\frac{1}{4}}=\frac{4}{21}$, which is *less* than the true value of the fraction. Hence, generally,

By stopping at an odd reduction, and neglecting the fractional part, the result is too great; but by stopping at an even reduction, and neglecting the fractional part, the result is too small.

Since $\frac{2}{7}=\cfrac{1}{3+\frac{1}{2}}$, we find

$$\frac{30}{157}=\cfrac{1}{5+\cfrac{1}{4+\cfrac{1}{3+\frac{1}{2}}}}$$

1st	reduction,	too great;
2d	"	too small;
3d	"	too great;
4th.	"	true value.

By this process we find

$$\frac{13}{30}=\cfrac{1}{2+\cfrac{1}{3+\frac{1}{4}}} \qquad \frac{49}{204}=\cfrac{1}{4+\cfrac{1}{6+\frac{1}{8}}}$$

349. The different quantities

$$\frac{1}{a},\quad \frac{1}{a+\frac{1}{b}},\quad \frac{1}{a+\frac{1}{b+\frac{1}{c}}},\ \text{etc}\ ,$$

are called *converging fractions*, because each one in succession gives a nearer value of the given expression.

The fractions $\frac{1}{a}$, $\frac{1}{b}$, $\frac{1}{c}$, etc., are called *integral* fractions.

350. *To explain the manner in which the converging fractions are found from the integral fractions.*

1. $\frac{1}{a}$ . . . . $=\frac{1}{a}$ 1^{st} conv. fraction.

2. $\frac{1}{a+\frac{1}{b}}$ . . . . $=\frac{b}{ab+1}$ 2^{d} conv. fraction.

3. $\frac{1}{a+\frac{1}{b+\frac{1}{c}}}$ . . . $=\frac{bc+1}{c(ab+1)+a}$ 3^{d} conv. fraction.

By examining the third converging fraction, we find it is formed from the 1^{st}, and 2^{d}, and from the 3^{d} integral fraction, as follows:

Num. $=3^{d}$ quot.$\times$ num. of 2^{d} conv. fract.$+$num. of 1^{st} conv. fract.
Denom.$=3^{d}$ quot.$\times$den. of 2^{d} conv. fract.$+$den. of 1^{st} conv. fract.

To prove the general law of formation, let $\frac{P}{P'}$, $\frac{Q}{Q'}$, $\frac{R}{R'}$ be the three converging fractions corresponding to the three integral fractions $\frac{1}{a}$, $\frac{1}{b}$, and $\frac{1}{c}$, and, as has already been shown,

$$\frac{R}{R'}=\frac{Qc+P}{Q'c+P'}.$$

Let us now take the next integral fraction $\frac{1}{d}$, and let $\frac{S}{S'}$ express the 4^{th} converging fraction. Then, it is obvious that $\frac{R}{R'}$ will become $\frac{S}{S'}$ by substituting $c+\frac{1}{d}$, instead of c; hence,

$$\frac{S}{S'}=\frac{Q\left(c+\frac{1}{d}\right)+P}{Q'\left(c+\frac{1}{d}\right)+P'}=\frac{(Q\,c+P\,)d+Q}{(Q'c+P')d+Q'}=\frac{R\,d+Q}{R'd+Q'}.$$

From this we see that the same rule applies to the 4^{th} converging fraction, and so on. Hence, for the n^{th} converging fraction,

Multiply the denominator of the n^{th} *integral fraction by the numerator of the* $(n-1)^{th}$ *converging fraction, and add to the product the numerator of the* $(n-2)^{th}$ *converging fraction. This will give the numerator of the* n^{th} *converging fraction.*

Multiply the denominator of the n^{th} *integral fraction by the denominator of the* $(n-1)^{th}$ *converging fraction, and add to the product the denominator of the* $(n-2)^{th}$ *converging fraction. This will give the denominator of the* n^{th} *converging fraction.*

Ex.—To find a series of converging fractions for $\frac{84}{227}$.

The integral fractions are $\frac{1}{2}$, $\frac{1}{1}$, $\frac{1}{2}$, $\frac{1}{2}$, $\frac{1}{1}$, $\frac{1}{3}$, $\frac{1}{2}$.

The converging fractions are $\frac{1}{2}$, $\frac{1}{3}$, $\frac{3}{8}$, $\frac{7}{19}$, $\frac{10}{27}$, $\frac{37}{100}$, $\frac{84}{227}$.

351. If the 2^d converging fraction (Art. 350) be subtracted from the 1^{st}, the remainder will be found to be a fraction having for its numerator $+1$, and for its denominator the product of the two denominators; and if the 3^d be subtracted from the 2^d, the resulting fraction will have -1 for its numerator, and the product of the denominators for its denominator.

By a process of reasoning similar to that employed in Art. 350, it may be shown, in a general manner, that

The difference between any two consecutive converging fractions is always a fraction having $+1$, *or* -1, *for its numerator, and the product of the two denominators for its denominator, according as the fraction subtracted is in an even or odd place.*

352. *To show that every converging fraction is in its lowest terms; and to find the approximate value of the fraction* $\frac{a}{b}$.

If $\frac{A}{B}$ and $\frac{C}{D}$ be any two consecutive converging fractions, by Art. 351, $\frac{A}{B}-\frac{C}{D}=+\frac{1}{BD}$, or $-\frac{1}{BD}$; that is, $AD-BC=\pm 1$.

Now, if A and B have a common divisor *greater* than 1, it will divide their multiples AD and BC, and their difference ± 1, (Art. 100); or, a quantity greater than 1 is a divisor of 1, which is impossible; hence, $\frac{A}{B}$ is in its lowest terms.

Since the denominators of the convergents continually increase, and their values continually diminish, and since the true value of $\frac{a}{b}$ lies between any two consecutive convergents, it is evident that by continuing the series, any degree of approximation to the true value may be obtained.

353. *To express* $\sqrt{N}$, *when* $N=a^2+1$, *in the form of a continued fraction.*

$$\sqrt{a^2+1}=a+\sqrt{a^2+1}-a=a+\frac{1}{\sqrt{a^2+1}+a} \text{ (Art. 206)},$$

$$=a+\frac{1}{a+a+\sqrt{a^2+1}-a} \quad =a+\cfrac{1}{2a+\cfrac{1}{2a+\cfrac{1}{2a}+}}, \text{ etc.}$$

Ex. $\sqrt{17}=\sqrt{4^2+1}=4+\cfrac{1}{8+\cfrac{1}{8+\cfrac{1}{8}+}}$, etc., the converging fractions to be added to 4, are $\frac{1}{8}$, $\frac{8}{65}$, $\frac{65}{528}$, etc.

354. To convert $\sqrt{N}$, where $N=a^2+b$, into a continued fraction.

Ex. Convert $\sqrt{19}$, or $\sqrt{16+3}$, into a continued fraction.

$$\sqrt{19}=4+\frac{1}{a}.$$

Hence, $$a=\frac{1}{\sqrt{19}-4}=\frac{\sqrt{19}+4}{3}=2+\frac{1}{b}.$$

$$b=\frac{3}{\sqrt{19}-2}=\frac{3(\sqrt{19}+2)}{15}=\frac{\sqrt{19}+2}{5}=1+\frac{1}{c}.$$

$$c=\frac{5}{\sqrt{19}-3}=\frac{5(\sqrt{19}+3)}{10}=\frac{\sqrt{19}+3}{2}=3+\frac{1}{d}.$$

Hence, . . . $$\sqrt{19}=4+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{3}+}}, \text{ etc.}$$

Proceeding in the same manner, the successive values of a, b, c, d, e, and f. will be found 2, 1, 3, 1, 2, 8. The value of g is the same as that of a, consequently, the succeeding values will recur in the same order as before.

The converging fractions are $\frac{4}{1}$, $\frac{9}{2}$, $\frac{13}{3}$, $\frac{48}{11}$, $\frac{61}{14}$, etc.

355. *To find the value of a continued fraction, when the denominators* q, r, s, *etc., of the integral fractions recur ad infinitum in a certain order.*

Ex. 1.—Let $\cfrac{1}{q+\cfrac{1}{r+\cfrac{1}{q+\cfrac{1}{r}}}}=x$, etc., ad infinitum.

Then, . . . $\cfrac{1}{q+\cfrac{1}{r+x}}=x$, or $\dfrac{r+x}{qr+qx+1}=x$.

From this equation, the value of x is easily found.

356. *To find in the form of a continued fraction the value of* x, *which satisfies the equation* $a^x=b$.

Ex.—Required the value of x in the equation $10^x=2$.

By substituting 0 and 1 for x, it appears that $x>0$ and <1.

Let . . $x=\dfrac{1}{y}$; then, $10^{\frac{1}{y}}=2$, or $2^y=10$.

Since $2^3=8$, and $2^4=16$, one of which is less and the other greater than 10; therefore, $y>3$, and <4; let $y=3+\dfrac{1}{z}$;

Then, ... $2^{3+\frac{1}{z}}=10$;

Or, . . . $2^3.2^{\frac{1}{z}}=10$, or $2^{\frac{1}{z}}=\frac{10}{8}=1.25$;

Therefore, ... $(1.25)^z=2$.

Again, it appears that $z>3$, and <4; let $z=3+\dfrac{1}{u}$; then, $(1.25)^{3+\frac{1}{u}}=(1.25)^3(1.25)^{\frac{1}{u}}=2$ .. $(1.25)^{\frac{1}{u}}=\dfrac{2}{(1.25)^3}=1.024$;

Therefore, $(1.024)^u=1.25$, and by trial $u>9$ and <10.

Hence, ... $x=\cfrac{1}{3+\cfrac{1}{3+\cfrac{1}{9}+}}$, etc.

This gives $x=\frac{1}{3}-$, $\frac{3}{10}+$, $\frac{28}{93}-=.30107$ nearly, etc.

Reduce each of the following to a continued fraction, and find the successive integral and converging fractions:

1. $\frac{130}{421}$ Ans. Integral fractions $\frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}$.
Converging " $\frac{1}{3}, \frac{4}{13}, \frac{21}{68}, \frac{130}{421}$.

2. $\frac{130}{291}$ Ans. Integral fractions $\frac{1}{2}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}$.
Converging " $\frac{1}{2}, \frac{4}{9}, \frac{21}{47}, \frac{130}{291}$.

3. $\frac{157}{972}$ Ans. Integral fractions $\frac{1}{6}, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}$.
Converging " $\frac{1}{6}, \frac{5}{31}, \frac{21}{130}, \frac{68}{421}, \frac{157}{972}$.

4. The height of Mt. Etna is 10963 feet, of Vesuvius 3900 feet; required the approximate ratio of the height of the former to that of the latter.

Ans. $\frac{1}{2}, \frac{1}{3}, \frac{5}{14}, \frac{16}{45}, \frac{37}{104}, \frac{90}{253}, \frac{127}{357}, \frac{3900}{10963}$.

5. The height of Mt. Perdu, the highest of the Pyrenees, is 11283 feet; that of Mt. Hecla is 4900 feet; required the approximate ratio of the height of the former to that of the latter. Ans. $\frac{1}{2}, \frac{3}{7}, \frac{10}{23}, \frac{33}{76}, \frac{76}{175}$, etc.

6. When the diameter of a circle is 1, the circumference is found to be greater than 3.1415926, and less than 3.1415927; required the series of fractions converging to the ratio of the circumference to the diameter.

Ans. $\frac{1}{3}, \frac{7}{22}, \frac{106}{333}$, and $\frac{113}{355}$.

Show that this last ratio, $\frac{113}{355}$, is true to within less than three ten-millionths of the circumference.

Suggestion.—In examples of this kind the integral fractions, corresponding to both fractions, should be found, and then the converging fractions calculated from those integral fractions that are the same in both series.

7. Express approximately the ratio of 24 hr. to 5 hr., 48 min., 49 sec., the excess of the solar yr. above 365 da.

Ans. $\frac{1}{4}, \frac{7}{29}, \frac{8}{33}, \frac{31}{128}, \frac{39}{161}, \frac{655}{2704}, \frac{694}{2865}, \frac{1349}{5569}, \frac{20929}{86400}$.

Hence, after every 4 years, we must have had 1 intercalary day, as in leap year; after every 29 years, we ought to have had 7 intercalary days; after every 33 years we ought to have had 8 inter-

calary days. This last was the correction used by the Persian astronomers, who had 7 regular leap years, and then deferred the eighth until the fifth year, instead of having it on the fourth.

8. Find the least fraction with only two figures in each term, approximating to $\frac{1947}{3359}$. Ans. $\frac{11}{19}$.

9. The lunar month, calculated on an average of 100 years, is 27.321661 days. Find a series of common fractions approximating nearer and nearer to this quantity.

Ans. $\frac{27}{1}$, $\frac{82}{3}$, $\frac{765}{28}$, $\frac{3907}{143}$, etc.

10. Find a series of fractions converging to $\sqrt{2}$.

Ans. $\frac{1}{1}$, $\frac{3}{2}$, $\frac{7}{5}$, $\frac{17}{12}$, $\frac{41}{29}$, etc.

11. Show that $\sqrt{5}$ is $> \frac{682}{305}$, and $< \frac{2889}{1292}$.

12. If $8^x=32$, find x. . . . . Ans. $\frac{5}{3}$.

13. If $3^x=15$, find x. . . . . . . Ans. 2.465.

LOGARITHMS

357. This method of computation was invented by *Lord Napier*, but subsequently much improved by *Mr. Henry Briggs*, whose system is now universally adopted in numerical computations.

The advantage, secured in the use of logarithms, arises from the application of the *law of exponents*, by which multiplication, division, involution, and evolution are performed by addition, subtraction, multiplication, and division.

Thus, $a^5\times a^2=a^7$, $\frac{a^7}{a^3}=a^4$, $(a^3)^5=a^{15}$, $\sqrt[5]{a^{20}}=a^4$.

If some number, arbitrarily assumed, be taken as a *base*, then

The LOGARITHM *of any number is the exponent of that power of the base, which is equal to that number.*

Thus, if a is the base of a system of logarithms, N, N′, N″, etc., any numbers, and

$$a^2=N,\ a^3=N',\ a^x=N'',$$

then, 2, 3, and x are called the logarithms of N, N′, and N″, in the system whose base is a.

The base of "Briggs' Logarithms," or the common system, is the number 10. Assuming this, we shall have

$(10)^0=1$	; hence, 0 is the log. of 1;				
$(10)^1=10$	;	"	1 " "	log. of 10;	
$(10)^2=100$	;	"	2 " "	log. of 100;	
$(10)^3=1000$	;	"	3 " "	log. of 1000;	
$(10)^4=10000$	;	"	4 " "	log. of 10000;	
Etc.,				Etc.	

The logarithm of every number between 1 and 10 is, evidently, some number between 0 and 1; that is, a proper fraction.

The logarithm of every number between 10 and 100 is some number between 1 and 2: that is, 1 plus a fraction.

The logarithm of every number between 100 and 1000 is 2 plus a fraction; and so on.

358. The integral part of a logarithm is called the *index* or *characteristic* of the logarithm.

Since the logarithm of 1 is 0, of 10 is 1, of 100 is 2, of 1000 is 3, and so on; therefore, for any number greater than unity,

The **Characteristic** *of the logarithm is one less than the number of integral figures in the given number.*

Thus, the logarithm of 123 is 2 plus a fraction; the logarithm of 1234 is 3 plus a fraction, and so on.

359. The computation of logarithms, in the common system, consists in finding the values of x in the equation

$$10^x=N, \text{ when N is successively } 1,\ 2,\ 3, \text{ etc.}$$

One method of finding an approximate value of x has been explained in Art. 356, but other methods more expeditious will be given hereafter.

The following table contains the logarithms of numbers from 1 to 100 in the common system:

N.	Log.	N.	Log.	N.	Log.	N.	Log.
1	0.000000	26	1.414973	51	1.707570	76	1.880814
2	0.301030	27	1.431364	52	1.716003	77	1.886491
3	0.477121	28	1.447158	53	1.724276	78	1.892095
4	0.602060	29	1.462398	54	1.732394	79	1.897627
5	0.698970	30	1.477121	55	1.740363	80	1.903090
6	0.778151	31	1.491362	56	1.748188	81	1.908485
7	0.845098	32	1.505150	57	1.755875	82	1.913814
8	0.903090	33	1.518514	58	1.763428	83	1.919078
9	0.954243	34	1.531479	59	1.770852	84	1.924279
10	1.000000	35	1.544068	60	1.778151	85	1.929419
11	1.041393	36	1.556303	61	1.785330	86	1.934498
12	1.079181	37	1.568202	62	1.792392	87	1.939519
13	1.113943	38	1.579784	63	1.799341	88	1.944483
14	1.146128	39	1.591065	64	1.806180	89	1.949390
15	1.176091	40	1.602060	65	1.812913	90	1.954243
16	1.204120	41	1.612784	66	1.819544	91	1.959041
17	1.230449	42	1.623249	67	1.826075	92	1.963788
18	1.255273	43	1.633468	68	1.832509	93	1.968483
19	1.278754	44	1.643453	69	1.838849	94	1.973128
20	1.301030	45	1.653213	70	1.845098	95	1.977724
21	1.322219	46	1.662758	71	1.851258	96	1.982271
22	1.342423	47	1.672098	72	1.857333	97	1.986772
23	1.361728	48	1.681241	73	1.863323	98	1.991226
24	1.380211	49	1.690196	74	1.869232	99	1.995635
25	1.397940	50	1.698970	75	1.875061	100	2.000000

In works on Trigonometry, Surveying, etc., where a set of logarithmic tables is given, the *characteristic* is usually omitted, and must be supplied by the rule given in Art. 358.

360. General Properties of Logarithms.—Let N and N′ be any two numbers, x and x' their respective logarithms, and a the base of the system; or, take any two numbers in the common system. Then, (Art. 357),

$$10^5=100000, \qquad a^x=N \quad . \quad . \quad . \ (1),$$

$$10^2=100, \qquad a^{x'}=N' \qquad (2).$$

Multiplying equations (1) and (2) together, we find

$$10^7 = 10000000, \quad . \quad . \quad . \quad . \quad . \qquad = a^{x+x'} = NN'.$$

But, by the definition of logarithms, 7 and $x+x'$ are the logarithms of 10000000 and of NN' respectively. Hence,

The sum of the logarithms of two numbers is equal to the logarithm of their product.

Similarly, the sum of the logarithms of three or more factors, is equal to the logarithm of their product. Hence, to multiply two or more numbers by means of logarithms,

Rule.—*Add together the logarithms of the numbers for the logarithm of the product.*

361. Taking the same equations, (Art. 360), we have

$$10^5 = 100000, \qquad a^x = N \quad . \; . \; . \; (1),$$
$$10^2 = 100, \qquad a^{x'} = N' \; . \; . \; . \; . \; (2).$$

Dividing equation (1) by equation (2), we find

$$10^3 = 1000, \quad . \quad . \quad . \quad . \quad a^{x-x'} = \frac{N}{N'}.$$

But, by the definition of logarithms, 3 and $x-x'$ are the logarithms of 1000 and of $\frac{N}{N'}$. Hence, to divide by means of logarithms,

Rule.—*From the logarithm of the dividend subtract the logarithm of the divisor for the logarithm of the quotient.*

1. Find the product of 9 and 6 by means of logarithms.

By the table (page 326), the log. of 9 is . .	0.954243
" " the log. of 6 is . .	0.778151
The sum of these logarithms is . .	1.732394

and the number corresponding in the table is 54.

2. Find the quotient of 63 divided by 9, by means of logarithms.

The log of 63 is	1.799341
log of 9 is . . .	0.954243
The difference is . . .	0.845098

and the number corresponding to this log. is 7.

By means of logarithms

3. Find the product of 7 and 8.
4. Find the continued product of 2, 3, and 7.
5. Find the quotient of 85 divided by 17.
6. Find the quotient of 91 divided by 13.

362. Resuming equation (1), (Art. 360), we have

$$10^2=100, \quad . \quad . \quad . \quad . \quad a^x=\text{N}.$$

Raising both sides to the 3d and to the m^{th} power, we find

$$10^6=1000000, \qquad a^{mx}=\text{N}^m.$$

But, (Art. 357), 6 and mx are the logarithms of 1000000 and of N^m respectively. Hence, to raise a number to any power by means of logarithms,

Rule.—*Multiply the logarithm of the given number by the exponent of the required power for the logarithm of the power of the number.*

363. Take the same equation

$$10^6=1000000, \quad . \quad . \quad . \quad . \quad . \quad a^x=\text{N}.$$

Extracting the 3d and n^{th} root of both sides, we have

$$10^2=100, \quad . \quad . \quad . \quad a^{\frac{x}{n}}=\text{N}^{\frac{1}{n}}.$$

But, (Art. 357), 2 and $\frac{x}{n}$ are the logarithms of 100 and of $\text{N}^{\frac{1}{n}}$ respectively. Hence, to extract any root of a number,

Rule.—*Divide the logarithm of the given number by the index of the required root for the logarithm of the root of the number.*

1. Find the third power of 4 by means of logarithms.

The logarithm of 4 is	0.602060
Multiply by the exponent 3	3
The product is	1.806180

which is the logarithm of 64.

2. Extract the fifth root of 32 by means of logarithms.

The logarithm of 32 is	1.505150
Dividing by the index 5, the quotient is . . .	0.301030

which is the logarithm of 2, the required root.

Solve the following examples by means of logarithms:

3. Find the square of 7.
4. Find the fourth power of 3.
5. Extract the cube root of 27.
6. Extract the sixth root of 64.

Other examples may be taken from arithmetic. It is, however, the province of algebra to explain the principles of logarithms, and the methods of computing the tables, rather than their use in actual calculations.

364. By means of *negative* exponents, we can also express the logarithms of fractions less than 1. Thus, in the common system, since

$(10)^{-1}=\frac{1}{10}=.1$, therefore, -1 is the log. of $.1$;
$(10)^{-2}=\frac{1}{100}=.01$, " -2 " log. of $.01$;
$(10)^{-3}=\frac{1}{1000}=.001$, " -3 " log. of $.001$;
$(10)^{-4}=\frac{1}{10000}=.0001$, " -4 " log. of $.0001$;
Etc., Etc.

The logarithm of any fraction between one and one-tenth, between one-tenth and one-hundredth, etc., may be expressed thus,

$$\text{Log. }(\tfrac{7}{10})=\text{log. }(\tfrac{1}{10}\times 7)=\text{log. }\tfrac{1}{10}+\text{log. }7=-1+\text{log. }7.$$
$$\text{Log. }(\tfrac{3}{100})=\text{log. }(\tfrac{1}{100}\times 3)=\text{log. }\tfrac{1}{100}+\text{log. }3=-2+\text{log. }3.$$

It is customary not to perform the subtraction indicated, but to unite the logarithm of the numerator to the negative characteristic. Thus,

$$\text{Log. } 0.7 = -1 + \text{log. } 7 = -1.845098, \text{ or } \bar{1}.845098.$$
$$\text{Log. } 0.03 = -2 + \text{log. } 3 = -2.477121, \text{ or } \bar{2}.477121.$$
$$\text{Log. } 0.004 = -3 + \text{log. } 4 = -3.602060, \text{ or } \bar{3}.602060.$$

Hence, *the characteristic of the logarithm of a decimal fraction is a negative number, and is numerically equal to the distance of the first significant figure from the decimal point.*

365. On the principle above explained, we may deduce the following

General Rule for finding from the Tables the Logarithms of any Decimal Fraction.—1. *Find the logarithm of the figures composing the decimal as if the fraction were a whole number.*

2. *Prefix the negative characteristic according to the rule given in Art.* 364.

366. The following examples, illustrative of the principles already explained, will afford a useful exercise:

1. $\text{Log. } (a.b.c.d..) = \text{log. } a + \text{log. } b + \text{log. } c + \text{log. } d.$

2. $\text{Log. } \left(\dfrac{abc}{de}\right) = \text{log. } a + \text{log. } b + \text{log. } c - \text{log. } d - \text{log. } e.$

3. $\text{Log. } (a^m.b^n.c^p.) = m \text{ log. } a + n \text{ log. } b + p \text{ log. } c.$

4. $\text{Log. } \left(\dfrac{a^m.b^n}{c^p}\right) = m \text{ log. } a + n \text{ log. } b - p \text{ log. } c$

5. $\text{Log. } (a^2 - x^2) = \text{log. } [(a+x)(a-x)] = \text{log. } (a+x) +$ $\text{log. } (a-x).$

6. $\text{Log. } \sqrt{a^2 - x^2} = \frac{1}{2} \text{ log. } (a+x) + \frac{1}{2} \text{ log. } (a-x).$

7. Log. $(a^3 \times \sqrt[4]{a^3}) = 3\frac{3}{4}$ log. a.

8. Log. $\frac{\sqrt{a^2-x^2}}{(a+x)^2} = \frac{1}{2}\{\log. (a-x) - 3 \log. (a+x)\}$.

367. Let us resume the equation $a^x = N$.

1st. If we make $x=1$, we have $a^1 = N = a$; hence, log. $=1$; that is,

Whatever be the base of the system, its logarithm in that system is 1.

2d. If we make $x=0$, in the equation $a^x=N$, we have

$$a^0 = N = 1 \text{; hence, log. } 1 = 0 \text{; that is,}$$

In any system the logarithm of 1 *is* 0.

368. In the equation $a^x = N$, consider $a > 1$, as in the common and the Naperian systems, and x negative; we then have

$$a^{-x} = \frac{1}{a^x} = N, \text{ and } \frac{1}{a^\infty} = a^{-\infty} = 0, \text{ or log. } 0 = -\infty.$$

Hence, *the logarithm of* 0, *in a system whose base is greater than* 1, *is an infinite number and negative.*

In a similar manner, it may be shown that in a system whose base is *less* than 1, the logarithm of 0 is *infinite* and *positive.*

369. As the *positive* and *negative* characteristics are taken to designate *whole numbers* and *fractions*, there remains no method of designating negative quantities by logarithms; or, as N, in each of the equations $a^x=N$ and $a^{-x}=N$, is positive,

Negative numbers have no real logarithms.

COMPUTATION OF LOGARITHMS.

370. Before proceeding to explain the methods of computing logarithms, we may observe that *it is only necessary to compute the logarithms of the prime numbers.*

For, the logarithm of every *composite* number is equal to the sum of the logarithms of its factors. Hence, the logarithms of 1, 2, 3, 5, 7, etc., being known, we can find those of 4, 6, 8, etc. Thus,

$4=2^2$;	hence,	$\log. 4=2\log. 2$, (Art. 362);
$6=2\times3$;	"	$\log. 6=\log. 2+\log. 3$;
$8=2^3$;	"	$\log. 8=3\log. 2$;
$9=3^2$;	"	$\log. 9=2\log. 3$;
$10=2\times5$;	"	$\log. 10=\log. 2+\log. 5$.

1. Suppose the logarithms of the numbers 2, 3, 5, and 7 to be known; show how the logarithms of the composite numbers from 12 to 30 may be found.

2. Of what numbers between 30 and 100, may the logarithms be found from those of 2, 3, 5, and 7; and why?

Ans. Of 23 different numbers, from 32 to 98.

371. In the common system, the equation $a^x=N$ (Art. 357) becomes $10^x=N$.

If we multiply both sides by 10, we have

$$10^x\times10=10^{x+1}=10N;$$

Also, . . $$10^x\times100=10^x\times10^2=10^{x+2}=100N.$$

Hence, in the common system, the logarithm of any number will become the logarithm of 10 times, 100 times, etc., that number, by increasing the characteristic by 1, 2 etc. From this results the advantage of *Briggs' system.*

Thus, the log. of	3	is	0.477121,
" "	30	"	1.477121,
" "	300	"	2.477121.

Also, the log. of	.2583 is	-1.412124,
"	2.583 "	0.412124,
"	25.83 "	1.412124.

372. If we compare the different powers of 10 with their logarithms in the common system, we have

Numbers	1,	10,	100,	1000,	10000,	
Logarithms	0,	1,	2,	3,	4,	and so on.

Hence, while the numbers are in *geometrical progression*, their logarithms are in *arithmetical progression*.

Therefore, if we take a geometrical mean between two numbers, and an arithmetical mean between their logarithms, the latter number will be the logarithm of the former.

Thus, the geometrical mean between 10 and 1000 is $\sqrt{10\times 1000}=100$, and the arithmetical mean between their logarithms, 1 and 3, is $(1+3)\div 2=2$.

In general, if $a^x=\text{N}$, and $a^{x'}=\text{N}'$; then,

$$\text{Log. of } \sqrt{\text{NN}'} \text{ is } \frac{x+x'}{2}.$$

By means of this principle, the common, or Briggean, system of logarithms was originally calculated.

Ex.—Let it be required to calculate the logarithm of 5.

First.—The proposed number lies between 1 and 10; hence, its logarithm will lie between 0 and 1.

The geometrical mean is $\sqrt{(1\times 10)}=3.162277$; the arithmetical mean is $(0+1)\div 2=0.5$. Hence, the log. of 3.162277 is 0.5.

Secondly.—Take the numbers 3.162277 and 10, and their logarithms .5 and 1, we find

The log of 5.623413 is 0.75.

Thirdly.—Take the numbers 3.162277 and 5.623413, and their logarithms 0.5 and 0.75, we find

The log. of 4.216964 is 0.625.

Fourthly.—Take the numbers 4.216964 and 5.623413, and their logarithms 0.625 and 0.75, we find

The log. of 4.869674 is 0.6875.

By continuing this process, always taking the two numbers nearest to 5, one of which is *less* and the other *greater*, after twenty-two operations, we obtain the number 5.000000+, and its corresponding logarithm 0.698970+.

Having the log. of 5 we readily find that of 2, or $\frac{10}{5}$ (Art. 361).

To find the log. of 3, take the numbers 2 and 3.162277, and their logarithms, and proceed as in finding the log. of 5.

373. Logarithmic Series.—The most convenient method of computing logarithms is by means of *Series*, which we shall now proceed to explain.

Let x be a number whose logarithm is to be expressed in a series, and let us apply the method of Indeterminate Coëfficients (Art. 314).

If we assume log. $x=A+Bx+Cx^2+Dx^3+$, etc., and make $x=0$, we have, log. $0=A=\infty$ (Art. 368). Hence,

$$\infty=A, \text{ which is absurd.}$$

If we assume log. $x=Ax+Bx^2+Cx^3+$, etc., and make $x=0$, we have log. $0=0$; that is, (Art. 368), $\infty=0$, which is also absurd. Hence, it is impossible to develope the logarithm of a number in powers of that number.

But if we assume

$$\text{Log. }(1+x)=Ax+Bx^2+Cx^3+Dx^4+, \text{ etc.} \quad . \quad . \quad (1)$$

and make $x=0$, we have log. $1=0$, which is correct (Art. 367).

In like manner, also assume

$$\text{Log. }(1+z)=Az+Bz^2+Cz^3+Dz^4+, \text{ etc.} \quad . \quad . \quad . \quad (2)$$

Subtracting equation (2) from (1) we get

$$\text{Log. }(1+x)-\text{log. }(1+z)=A(x-z)+B(x^2-z^2)$$
$$+C(x^3-z^3)+, \text{ etc.} \qquad (3).$$

The second member of this equation is divisible by $x-z$ (Art. 83); let us reduce the first member to a form in which it shall also be divisible by the same factor. By Art. 361,

$$\text{Log. } (1+x) - \log. (1+z) = \log. \left(\frac{1+x}{1+z}\right) = \log. \left(1+\frac{x-z}{1+z}\right).$$

Now, regarding $\frac{x-z}{1+z}$ as a single quantity, we may assume

$$\text{Log. } \left(1+\frac{x-z}{1+z}\right) = \text{A}.\frac{x-z}{1+z} + \text{B}\left(\frac{x-z}{1+z}\right)^2 + \text{C}\left(\frac{x-z}{1+z}\right)^3 +, \text{ etc.}$$

Substituting this for log. $(1+x)$— log. $(1+z)$, in equation (3), and dividing both sides by $x-z$, we obtain

$$\text{A}.\frac{1}{1+z} + \text{B}.\frac{x-z}{(1+z)^2} + \text{C}.\frac{(x-z)^2}{(1+z)^3} +, \text{ etc.},$$
$$= \text{A} + \text{B}(x+z) + \text{C}(x^2+xz+z^2) +, \text{ etc.}$$

Since this equation is true for all values of x and z, it must be true when $x=z$. Making this supposition, we have

$$\text{A}.\frac{1}{1+x} = \text{A} + 2\text{B}x + 3\text{C}x^2 + 4\text{D}x^3 + 5\text{E}x^4 +, \text{ etc.};$$

or, performing the division of 1 by $1+x$, we have

$$\text{A}(1-x+x^2-x^3+x^4- \ldots) = \text{A} + 2\text{B}x + 3\text{C}x^2 + 4\text{D}x^3 +.$$

Equating the coëfficients of the like powers of x (Art. 314),

$$\text{A}=\text{A}, \quad \text{B}=-\frac{\text{A}}{2}, \quad \text{C}=\frac{\text{A}}{3}, \quad \text{D}=-\frac{\text{A}}{4}.$$

The law of this series is obvious, the coëfficient of the n^{th} term being $\pm\frac{\text{A}}{n}$, according as n is *odd* or *even*.

$$\text{Hence, log. } (1+x) = \text{A}x - \frac{\text{A}}{2}x^2 + \frac{\text{A}}{3}x^3 - \frac{\text{A}}{4}x^4 + \ldots$$
$$= \text{A}\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \ldots\right) \quad (4)$$

There still remains one quantity, A, undetermined. This is as it should be, for the logarithm of a given number is indeterminate unless the base of the system be given.

The value of A depends on the base of the system, so that when A is given, the base may be determined; or, when the base is known, A may be determined.

If we denote the series in the parenthesis in equation (4) by x', we may write

$$\text{Log. } (1+x)=\text{A}x'. \quad \text{Hence,}$$

The logarithm of a number consists of two factors, one of which depends on the number itself, and the other on the base of the system in which the logarithm is taken.

That factor which depends on the base is called the MODULUS *of the system of logarithms.*

Lord Napier, the inventor of logarithms, assumed the modulus equal to unity, and the system resulting from such a modulus, is called the *Naperian*, or *Hyperbolic system.*

For all values of x above $x=1$ the series (5) *diverges*, and is, therefore, inapplicable.

Designating the logarithms in this system by log'., we have

$$\text{Log}'.\ (1+x)=\frac{x}{1}-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+, \text{ etc.} \quad (5)$$

Thus, if $x=0$, we find log'. 1=0, as in Art. 367.

If we make $x=1$, we have

$$\text{Log}'.\ 2=1-\tfrac{1}{2}+\tfrac{1}{3}-\tfrac{1}{4}+\tfrac{1}{5}-, \text{ etc.}$$

374. The preceding series converges so slowly that it would be necessary to take a great number of terms to obtain a near approximation. But we may obtain a more converging series in the following manner:

Resuming equation (5),

$$\text{Log}'.\ (1+x)=\frac{x}{1}-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\frac{x^5}{5}-, \text{ etc.} \ldots \quad (5).$$

Substituting $-x$ for x, in this equation, we obtain

$$\text{Log}'.\ (1-x)=-\frac{x}{1}-\frac{x^2}{2}-\frac{x^3}{3}-\frac{x^4}{4}-\frac{x^5}{5}-, \text{ etc.} \ldots (6).$$

Subtracting equation (6) from (5), and observing that

$$\text{Log}'.\ (1+x)-\ \log'.\ (1-x)=\ \log'.\left(\frac{1+x}{1-x}\right), \text{ we have}$$

$$\text{Log}'.\ \frac{1+x}{1-x}=2\left(\frac{x}{1}+\frac{x^3}{3}+\frac{x^5}{5}+\frac{x^7}{7}+\frac{x^9}{9}+\ .\ .\ .\ \right).$$

Since $\frac{1+x}{1-x}=1+\frac{2x}{1-x}$, let $\frac{1+x}{1-x}=1+\frac{1}{z}$, $\therefore\ x=\frac{1}{2z+1}$,

$$\text{and } \log'.\ \frac{1+x}{1-x}=\ \log'.\left(1+\frac{1}{z}\right)=\ \log'.\left(\frac{z+1}{z}\right)$$

$$=\ \log'.\ (z+1)-\ \log'.\ z.$$

By substitution, the preceding series becomes

$$\text{Log}'.\ (z+1)-\ \log'.\ z=2\left\{\frac{1}{2z+1}+\frac{1}{3(2z+1)^3}+\frac{1}{5(2z+1)^5}+\ .\ .\ \right\};$$

$$\text{Log}'.\ (z+1)=\ \log'.\ z+2\left\{\frac{1}{2z+1}+\frac{1}{3(2z+1)^3}+\frac{1}{5(2z+1)^5}+\ .\ .\right\}\ (7).$$

375. By means of this series, the Naperian logarithm of any number may be computed, when the logarithm of the *preceding number* is known. But the log′. of 1 is 0, (Art. 367); therefore, making $z=1, 2, 4, 6$, etc., we obtain the following

NAPERIAN, OR HYPERBOLIC LOGARITHMS.

$$\text{Log}'.\ 2=\log'.\ 1+2\left\{\frac{1}{3}+\frac{1}{3\cdot 3^3}+\frac{1}{5\cdot 3^5}+\frac{1}{7\cdot 3^7}+\ .\ .\right\}=0.693147$$

$$\text{Log}'.\ 3=\log'.\ 2+2\left\{\frac{1}{5}+\frac{1}{3\cdot 5^3}+\frac{1}{5\cdot 5^5}+\frac{1}{7\cdot 5^7}+\ .\ .\right\}=1.098612$$

$$\text{Log}'.\ 4=2.\ \log.\ 2\ \ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ =1.386294$$

$$\text{Log}'.\ 5=\log'.\ 4+2\left\{\frac{1}{9}+\frac{1}{3\cdot 9^3}+\frac{1}{5\cdot 9^5}+\frac{1}{7\cdot 9^7}+\ .\ .\right\}=1.609438$$

$$\text{Log}'.\ 6=\log'.\ 2+\ \log'.\ 3\ \ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ =1.791759$$

$$\text{Log}'.\ 7=\log'.\ 6+2\left\{\frac{1}{13}+\frac{1}{3\cdot 13^3}+\frac{1}{5\cdot 13^5}+\ \ .\ \ .\ \ \right\}=1.945910$$

Log$'$. $8=3$ log$'$. 2, or log$'$. 2$+$ log$'$. 4 . . . . . $=2.079442$
Log$'$. $9=2$ log$'$. 3 . . . . $=2.197225$
Log$'$. $10=$ log$'$. 2$+$ log$'$. 5 . . $=2.302585$

In this manner the Naperian logarithms of all numbers may be computed.

When the numbers are large, their logarithms are computed more easily than in the case of small numbers. Thus, in calculating the logarithm of 101, the first term of the series gives the result true to seven places of decimals.

376. *To explain the method of computing common logarithms from Naperian logarithms.*

We have already found (Art. 373, Equation 4),

$$\text{Log. }(1+x)=\text{A}\left(\frac{x}{1}-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\frac{x^5}{5}-\frac{x^6}{6}+. \quad . \ .\right)$$

Denoting the Naperian logarithm by an accent, we have

$$\text{Log}'.\ (1+x)=\text{A}'\left(\frac{x}{1}-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\frac{x^5}{5}-\frac{x^6}{6}+. \quad .\ \right)$$

Since the series in the second members are the same, we have

$$\text{Log. }(1+x) : \text{log}'.\ (1+x) : : \text{A} . \text{A}'. \quad \text{Therefore,}$$

The logarithms of the same number, in two different systems, are to each other as the moduli of those systems.

But in Napier's system the modulus $\text{A}'=1$. Therefore,

$$\text{Log. }(1+x)=\text{A log}'.\ (1+x). \quad \text{Hence,}$$

To find the common logarithm of any number, multiply the Naperian logarithm of the number by the modulus of the common system.

It now remains to find the modulus of the common system.

From the equation, log. $(1+x)=\text{A}.$ log$'$. $(1+x)$,

We find . . . . . . $\text{A}=\dfrac{\text{log. }(1+x)}{\text{log}'.\ (1+x)}$. Hence,

The modulus of the common system is equal to the common log. of any number divided by the Naperian log. of the same number.

But the common logarithm of 10 is 1, and we have calculated the Naperian logarithm of 10, (Art. 375); therefore,

$$A=\frac{\log.\ 10}{\log'.\ 10}=\frac{1}{2.302585}=.4342944,$$

which is the modulus of the common system.

Hence, if N is any number, we have

$$\text{Com. log. N}=.4342944\times\ \text{Nap. log. N.}$$

On account of the importance of the number A, its value has been calculated with great exactness. It is

$$A=.4342944819032518276 5.$$

377. *To calculate the common logarithms of numbers directly.*

Having found the modulus of the common system, if we multiply both members of equation (7), Art. 374, by A, and recollect that $A\times$ Nap. log. N= com. log. N, the series becomes

$$\text{Log.}\ (z+1)=\log.\ z+2A\left\{\frac{1}{2z+1}+\frac{1}{3(2z+1)^3}+\frac{1}{5(2z+1)^5}+\ .\quad\right\}.$$

Or, by changing z into P, for the sake of distinction, and putting B, C, D, etc., to represent the terms immediately preceding those in which they are used, we have

$$\text{Log.}\ (P+1)=\log.\ P+\frac{2A}{2P+1}+\frac{B}{3(2P+1)^2}+\frac{3C}{5(2P+1)^2}$$

$$+\frac{5D}{7(2P+1)^2}+\frac{7E}{9(2P+1)^2}+\frac{9F}{11(2P+1)^2}+,\ \text{etc.}$$

We shall now exemplify its use in finding the logarithm of 2.

Here, P=1, and 2P+1=3.

$$\text{Log. P} = \log\ 1 \ldots\ldots\ldots = .00000000;$$

$$\frac{2A}{2P+1} = \frac{.86858896}{3} \ldots\ldots = .28952965; \quad (B.)$$

$$\frac{B}{3(2P+1)^2} = \frac{.28952965}{3\times3^2} \ldots\ldots = .01072332; \quad (C.)$$

$$\frac{3C}{5(2P+1)^2} = \frac{3\times.01072332}{5\times3^2} \ldots\ldots = .00071489; \quad (D.)$$

$$\frac{5D}{7(2P+1)^2} = \frac{5\times.00071489}{7\times3^2} \ldots\ldots = .00005674; \quad (E.)$$

$$\frac{7E}{9(2P+1)^2} = \frac{7\times.00005674}{9\times3^2} \ldots\ldots = .00000490; \quad (F.)$$

$$\frac{9F}{11(2P+1)^2} = \frac{9\times.00000490}{11\times3^2} \ldots\ldots = .00000045; \quad (G.)$$

$$\frac{11G}{13(2P+1)^2} = \frac{11\times.00000045}{13\times3^2} \ldots = .00000004; \quad (H.)$$

Therefore, common logarithm of 2 $= .30102999.$

EXERCISE.—In a similar manner let the pupil calculate the common logarithms of 3, 5, 7, and 11.

For the results to 6 places of decimals, see the Table, page 326.

378. *To find the base of the Naperian system of logarithms.*

If we designate the base by e, we have, (Art. 376),

$$\text{Log. } e : \log'. e :: A . A'.$$

But $A = .4342944$, $A' = 1$, and $\log'. e = 1$, (Art. 367); hence,

$$\text{Log. } e : 1 :: .4342944 : 1;\ \text{whence, } \log. e = .4342944.$$

Taking the number of which the logarithm is .4342944, from the table of common logarithms, we find $e = 2.71828182$.

We thus see that in both the common and the Naperian systems of logarithms, the base is *greater* than unity.

Napier's logarithms are used in the Calculus, but not in the common operations of multiplication, division, etc.

379. The student may prove the following theorems:

1. No system of logarithms can have a negative base, or have unity for its base.

2. The logarithms of the same numbers in two different systems have the same ratio to each other.

3. The difference of the logarithms of two consecutive numbers is less as the numbers themselves are greater.

SINGLE AND DOUBLE POSITION.

NOTE.—This subject is introduced in connection with that of logarithms, because the rule for Double Position is applied to the solution of exponential equations.

380. Single Position.—The Rule of Single Position is applied to the solution of questions which give rise to an equation of the form

$$ax = m \qquad (1).$$

If we assume x' to be the value of x, and denote by m' the result of the substitution of x' for x, we have

$$ax' = m' \qquad (2).$$

Comparing equations (1) and (2), we have

$$m' : m :: ax' : ax : x' : x; \text{ that is,}$$

As the result of the supposition is to the result in the question, so is the supposed number to the number required.

EXAMPLE.—What the number, whose third, fourth, and sixth part being added, the sum will be 45? Ans. 60.

381. Double Position.—In Double Position, the result, although it is dependent on the unknown quantity, does not increase or diminish in the same ratio with it.

The class of questions to which it is particularly applicable, gives rise to an equation of the form

$$ax+b=m \qquad (1).$$

If we suppose x' and x'' to be near values of x, and e' and e'' to be the errors, or the differences between the true result and the results obtained by substituting x' and x'' for x, we have

$$ax'+b=m+e' \qquad (2),$$
$$ax''+b=m+e'' \qquad (3).$$

If we subtract equation (1) from (2), and (3) from (2), we have

$$a(x'-x\ )=e' \qquad (4),$$
$$a(x'-x'')=e'-e'' \qquad (5).$$

From these equations, we easily obtain

$$\frac{x'-x''}{e'-e''}=\frac{x'-x}{e'} \qquad (6).$$

By subtracting equation (1) from (3), we also find

$$a(x''-x)=e'', \text{ and thence,}$$
$$\frac{x'-x''}{e'-e''}=\frac{x''-x}{e''} \qquad (7).$$

Hence, (Art. 263), *The difference of the errors is to the difference of the two assumed numbers, as the error of either result is to the difference between the true result and the corresponding assumed number.*

When the question gives rise to an equation of the form $ax+b=m$, this rule gives a result absolutely correct; but when the equation is of a less simple form, as in exponential equations (Art. 383), the result obtained is only approximately true.

Corollary.—The common arithmetical rule is deduced from the following value of x, found either from equation (6) or (7):

$$x=\frac{e'x''-e''x'}{e'-e''}.$$

EXPONENTIAL EQUATIONS.

382. An **Exponential Equation** is an equation in which the unknown quantity appears in the form of an exponent or index; as,

$$a^x=b,\ x^x=a,\ a^{bx}=c, \text{ etc.}$$

Such equations are most easily solved by means of logarithms.

Thus, in the equation . . . $a^x=b$,

We have (Art. 362), . . $x \log. a = \log. b$;

Or, ... $x=\frac{\log. b}{\log. a}$.

Ex. 1.—What is the value of x in the equation $2^x=64$?

Here, ... $x \log. 2 = \log. 64$;

Whence, . . . $x=\frac{\log. 64}{\log. 2}=\frac{1.806180}{.301030}=6$, Ans.

383. If the equation is of the form $x^x=a$, the value of x may be found by Double Position, as follows:

Find by *trial* two numbers nearly equal to the value of x; substitute them for x in the given equation, and note the results. Then, from (7), we have (Art. 263) the proportion,

As the difference of the errors, is to the difference of the two assumed numbers, so is the error of either result, to the correction to be applied to the corresponding assumed number.

Ex. 1.—Given $x^x=100$, to find the value of x.

The value of x is evidently between 3 and 4, since $3^3=27$, and $4^4=256$; hence, taking the logarithms of both sides,

$$x \log. x = \log. 100 = 2.$$

By trial, we readily find that x is greater than 3.5, and less than 3.6; then, let us assume 3.5 and 3.6 for the two numbers.

First Supposition.		*Second Supposition.*	
$x=3.5$;	log. $x=.544068$	$x=3.6$;	log. $x=.556303$
multiply by 3.5, we find		multiply by 3.6, we find	
x log. x	$=1.904238$	x. log. x	$=2.002690$
true no.	$=2.000000$	true no.	$=2.000000$
error	$=-.095762$	error	$+\ .002690$

Diff. results : Diff. assumed nos. : : Error 2d result : Its cor.

.098452 : 0.1 : : .002690 : .00273

Hence, $3.6-.00273=3.59727$ nearly.

By trial we find that 3.5972 is less, and 3.5973 greater than the true value; and by repeating the operation with these numbers, we would find $x=3.5972849$ nearly.

2. Given $20^x=100$, to find x. Ans. $x=1.53724$.

3. Given $x^x=5$, to find x. Ans. $x=2.129372$.

4. How many places of figures will there be in the number expressing the 64^{th} power of 2? Ans. 20.

5. Given $a^{bx+d}=c$, to find x. Ans. $x=\dfrac{\text{log. } c-d.\ \text{log. } a}{b.\ \text{log. } a}$

6. Given $a^{mx}b^{nx}=c$, to find x.

$$\text{Ans. } x=\frac{\text{log. } c}{m.\ \text{log. } a+n.\ \text{log. } b}.$$

7. Given $x+y=a$, and $m^{x-y}=n$, to find x and y.
Ans. $x=\frac{1}{2}(a+\text{ log. } n\div\text{ log. } m)$, $y=\frac{1}{2}(a-\text{ log. } n\div\text{ log. } m)$.

8. Given $2^x.3^z=2000$, and $3z=5x$, to find the values of x and z.

$$\text{Ans. } x=\frac{3(3+\text{ log. } 2)}{3\text{ log. } 2+5\text{ log. } 3},\ z=\frac{5(3+\text{ log. } 2)}{3\text{ log. } 2+5\text{ log. } 3}.$$

9. Given $a^{2x}-2a^x=8$, to find x. Ans. $x=\frac{2\log. 2}{\log. a}$.

Suggestion.—This is a quadratic form, therefore complete the square.

10. Given $2^{2x}+2^x=12$, to find x. Ans. $x=1.58496$.

11. Given $a^x+\frac{1}{a^x}=b$, to find x.

$$\text{Ans. } x=\frac{\log. \frac{1}{2}(b\pm\sqrt{b^2-4})}{\log. a}.$$

12. Given $x^y=y^x$, and $x^3=y^2$, to find x and y.
Ans. $x=2\frac{1}{4}$, $y=3\frac{3}{8}$.

13. Given $(a^2-b^2)^{2(x-1)}=(a-b)^{2x}$, to find x.

$$\text{Ans. } x=1+\frac{\log. (a-b)}{\log. (a+b)}.$$

14. Given $(a^4-2a^2b^2+b^4)^{x-1}=(a-b)^{2x}(a+b)^{-2}$, to find x.

$$\text{Ans. } x=\frac{\log. (a-b)}{\log. (a+b)}.$$

15. Given $x^y=y^x$, and $x^p=y^q$, to find x and y.

$$\text{Ans. } x=\left(\frac{p}{q}\right)^{\frac{q}{p-q}},\quad y=\left(\frac{p}{q}\right)^{\frac{p}{p-q}}.$$

16. Given $3^{(x^2-4x+5)}=1200$, to find x.
Ans. $x=4.33$, or -0.33.

INTEREST AND ANNUITIES.

384. The solution of *all* questions in Interest and Annuities may be simplified, and also generalized, by means of algebraical formulæ; but certain problems in Compound Interest and Annuities may be very much abridged by the use of logarithms.

Let P= the principal, or sum at interest in dollars.

$r=$ the interest of 1 \$ for *one* year.

$t=$ the time in years that P draws interest.

A= the amount, at the end of t years.

385. Simple Interest.—Since tr represents the interest of 1 \$ for t years, and Ptr, the interest of P \$ for t years; therefore,

$$A=P+Ptr=P(1+tr). \quad . \quad . \quad . \qquad (1).$$

From this equation, any three of the quantities P, r, t, A, being given, the fourth may be found. Thus,

$$P=\frac{A}{1+tr}, \quad t=\frac{A-P}{Pr}, \quad r=\frac{A-P}{Pt}.$$

Examples may be given from any treatise of arithmetic.

386. Compound Interest.—Let $R=1+r$, the amount of 1 \$ for one year; then, R will be the principal for the second year; and since the amount, in each case, is proportional to the principal for the same time; therefore,

$1:R::R:$the amount of 1 \$ in 2 years$=R^2$.

$1:R:\ R^2\ R^3$, the amount of 1 \$ in 3 years.

And, in like manner, R^t is the amount of 1 \$ in t years.

The amount of P\$ will be P times the amount of 1 \$. Hence,

$$A=P.R^t=P(1+r)^t; \text{ whence,}$$

$$\text{Log. } A=\log. P+t. \log. (1+r) \qquad (1).$$

$$\text{Log. } P=\log. A-t. \log. (1+r) \qquad (2).$$

$$t=\frac{\log. A-\log. P}{\log. (1+r)} \qquad (3).$$

$$\text{Log. } (1+r)=\frac{\log. A-\log P}{t} \qquad (4).$$

Corollary 1.—The interest $=A-P=PR^t-P=P(R^t-1)$.

Corollary 2.—If the interest is paid *half-yearly*, $t=2t$, and $r=\frac{r}{2}$. Hence,

$$A=P\left(1+\frac{r}{2}\right)^{2t} (5). \text{ If paid } \textit{quarterly}, A=P\left(1+\frac{r}{4}\right)^{4t} (6).$$

Corollary 3.—From the equation $A=P.R^t$, we can readily find the time in which any sum, at compound interest, will amount to *twice*, *thrice*, or m times itself.

Thus, if $A=2P$; then, $2P=PR^t$ $\therefore$ $R^t=2$, and $t=\frac{\log. 2}{\log. R}$.

" if $A=3P$; then, $R^t=3$, and $t=\log. 3 \div \log. R$;

" if $A=mP$; then, $R^t=m$, and $t=\log. m \div \log. R$.

1. Let it be required to find the time in which any sum will double itself at 10 per cent. compound interest.

Here, $r=.10$, $R=1+r=1+.10=1.10$; hence,

$$t=\frac{\log. 2}{\log. R}=\frac{.301030}{.041393}=7.272 \text{ yrs., Ans.}$$

2. What is the amount of 1 \$ for 100 years at 6 per cent. per annum, compound interest? Ans. \$339.28.

3. How many figures will express the amount of \$1 for 1000 yr., at 6 % per annum, comp. int.? Ans. 26.

4. In how many yr. will any sum double itself at compound interest, at 5, 6, 7, and 8 % per annum respectively? Ans. 14.2066, 11.8956, 10.2447, 9.0064 yrs.

5. In what time, at compound interest, reckoning 5 % per annum, will \$10 amount to \$100? Ans. 47.19 yrs.

6. If \$P, at compound interest, amount to \$M in t years, what sum will amount to \$P at the end of t years?

387. The increase of the population of a country may be computed on the same principles as compound interest.

1. The population of the United States in 1790 was 3900000, and in 1840, 17000000. Required the average rate of increase for each 10 years.

Here, there are 5 periods of 10 years each. Hence, by comparing the quantities given, with those in equation (4), Art. 386, we have

$$A=17000000,\ P=3900000,\ \text{and}\ t=5.$$

Log. A, (see table, page 326), ...	7.230449
Log. P	6.591065
Divide by 5	5)0.639384
Log. $(1+r)$ 1.342	0.127877

Hence, $r=1.342-1=.342=34\frac{1}{5}$ per cent., Ans.

2. The population of England in 1820 was 11000000, and in 1830 about 13000000. What was the annual rate of increase, and in what time would the population be doubled? Ans. .016 per cent., and 41.49 yrs.

388. Compound Discount.—The present value of a sum P, due t years hence, reckoning compound interest, is easily obtained from Art. 386.

Let $P'=$ the present worth, then in t years, P' at compound interest, will amount to P; therefore,

$$P=P'(1+r)^t,\ \therefore\ P'=\frac{P}{(1+r)^t} \qquad (1).$$

Let D= Comp. Discount; then, $D=P-P'=P-\frac{P}{(1+r)^t}$. (2)

From equation (1), $\log. P'=\log. P-t\log. (1+r)$ (3).

Ex.—What is the compound discount on \$1000, due in 20 years, at 5 per cent.? Ans. \$623.11.

389. Annuities Certain.—An *Annuity* is a sum of money which is payable at equal intervals of time.

An annuity already commenced is said to be *in possession;* one commencing after a certain number of years has

elapsed, is called a *deferred* annuity, or an annuity in *reversion*.

An *annuity certain* is one limited to a certain number of years. A *life annuity* is one which terminates with the life of any person. A *perpetuity*, or *perpetual* annuity, is one which is unlimited in its duration.

All the computations relating to annuities are made according to compound interest.

390. *To find the amount of an annuity in any number of years, at compound interest.*

Let a denote the annuity, p the present value, m the amount; and r, R, t, the same as in the preceding articles.

The first annuity a, becomes due at the end of the year, and thus, in $t-1$ years, will amount to aR^{t-1} (Art. 386). The second and third annuities, due in 2 and 3 years, amount to aR^{t-2} and aR^{t-3}, and so on to the last, which is a.

Hence, the entire amount is the sum of a geometrical series, whose first term $=aR^{t-1}$, common ratio $=R$, and last term $=a$; therefore, by reversing the order of the terms, we have

$$m=a+aR+aR^2+aR^3+ \quad . \quad . \quad . \quad +aR^{t-2}+aR^{t-1}.$$

$$\therefore \text{ (Art. 297), } m=a\frac{R^t-1}{R-1}=a\frac{(1+r)^t-1}{r}.$$

If the annuity is to be received in *half-yearly* installments,

We have $$m=\frac{a}{2}.\frac{(1+\frac{1}{2}r)^{2t}-1}{\frac{1}{2}r}=a.\frac{(1+\frac{1}{2}r)^{2t}-1}{r}.$$

If *quarterly*, $$m=\frac{a}{4}.\frac{(1+\frac{1}{4}r)^{4t}-1}{\frac{1}{4}r}=a.\frac{(1+\frac{1}{4}r)^{4t}-1}{r}.$$

Corollary.—Similarly, the amount of a dollars placed out annually for t successive years, at compound interest, would be

$$m=aR+aR^2+aR^3+, \text{ etc., } =aR.\frac{R^t-1}{R-1}.$$

1. To what sum will an annuity of \$120 for 20 years amount at 6 per cent. per annum? Ans. \$4414.27.

2. Three children, A, B, C, who come of age at the end of a, b, c, years, are to have \$P divided among them, so that their shares being placed at compound interest, each shall receive, at coming of age, the same sum. Find the share of A, the youngest. Ans. $\frac{P}{1+R^{a-b}+R^{a-c}}$.

3. What would \$100, put out annually at compound interest, amount to in 10 years at 6 per cent.? Ans. \$1397.16.

391. *To find the present value of an annuity to be paid* t *years, at compound interest.*

Let p denote the present value of the annuity a; then, the amount of p\$ in t years $=pR^t$ (Art. 386), and the amount of the annuity a in the same time is (Art. 390) $a.\frac{R^t-1}{R-1}$; but these two amounts must be equal to each other; hence, we get

$$pR^t=a.\frac{R^t-1}{R-1}, \text{ and } p=a.\frac{R^t-1}{R^t(R-1)}=\frac{a}{R-1}\left(1-\frac{1}{R^t}\right).$$

Corollary.—If the annuity is to continue forever, t becomes infinite, $\frac{1}{R^t}$ vanishes, and we have $p=\frac{a}{R-1}=\frac{a}{r}$.

1. What is the present worth of an annuity of \$250, payable yearly for 30 yr. at 5 %? Ans. \$3843.1135.

2. What is the present worth of a perpetual annuity of \$600 at 6 % per annum? Ans. \$10000.

392. *To find the present value of an annuity in reversion; that is, an annuity which is to commence at the end of* n *years, and to continue* t *years.*

By Art. 391, the present value of the annuity for $n+t$ years, is

$$\frac{a}{\mathrm{R}-1}\left(1-\frac{1}{\mathrm{R}^{n+t}}\right), \text{ and for } n \text{ years, } \frac{a}{\mathrm{R}-1}\left(1-\frac{1}{\mathrm{R}^{n}}\right).$$

The difference of these two sums is the value in reversion; therefore,

$$p=\frac{a}{\mathrm{R}-1}\left(\frac{1}{\mathrm{R}^n}-\frac{1}{\mathrm{R}^{n+t}}\right)=\frac{a}{r\mathrm{R}^n}\left(1-\frac{1}{\mathrm{R}^t}\right).$$

If the annuity is payable *forever* after n years, we have

$$p=\frac{a}{r\mathrm{R}^n}.$$

1. What is the present value of an annuity of \$112.50, to commence at the end of 10 years, and to continue 20 years, at 4 % ? Ans. \$1032.877.

2. What is the present value of an annuity of \$1000, to commence at the end of 15 years and continue forever, at 6 % per annum? Ans. \$6954.40.

3. A debt of a\$, accumulating at compound interest, is discharged in n years, by equal annual payments of b\$; find the value of n. Ans. $n=\dfrac{\log. b - \log. (b-ra)}{\log. (1+r)}$.

4. A debt of \$8000, at 6 % compound interest, is discharged by eight equal annual payments. Required the annual payment. Ans. \$1288.286.

XII. GENERAL THEORY OF EQUATIONS.

393. From Art. 113, it is obvious that, as

$ax+b=0$, is an equation of the 1st degree,

$x^2+bx+c=0$, is an equation of the 2d degree,

$x^3+bx^2+cx+d=0$, is an equation of the 3d degree; so,

$x^n+\mathrm{A}x^{n-1}+\mathrm{B}x^{n-2}+\mathrm{C}x^{n-3}+\ldots\ldots+\mathrm{T}x+\mathrm{V}=0$, is, in general, an equation of the n^{th} degree.

The coëfficients, A, B, C, etc., may be positive or negative, integral or fractional; and any of them may be equal to zero.

If the coëfficient of the highest power of x is not unity, it may be made so by division.

394. A **Root** of an equation is a number, or quantity, such that being substituted for the unknown quantity, the equation will be verified.

Thus, 3 is a root in the equation $x^3+2x^2-14x-3=0$.

A **Function** of a quantity is any expression dependent on that quantity. Thus, $2x+3$ is a function of x;

$5x^2$, is a function of x;

$7x-3y^2$, is a function of x and y.

In a series, when the signs of two successive terms are *alike*, they constitute a *permanence*, when they are *unlike*, a *variation*.

Thus, in the polynomial, $-r-s+t+u$, the signs of the first and second terms constitute a permanence, of the second and third a variation, and of the third and fourth a permanence.

395. Proposition I.—*If* a *is a root of any equation,*

$$x^n+Ax^{n-1}+Bx^{n-2}+Cx^{n-3}+\ldots\ldots+Tx+V=0, \quad (n),$$

then will the equation be divisible by x—a.

For if a is one value of x, the equation will be verified when a is substituted for x. This gives

$$a^n+Aa^{n-1}+Ba^{n-2}+Ca^{n-3}+\ldots\ldots+Ta+V=0;$$

$$\text{Or,}\ V=-a^n-Aa^{n-1}-Ba^{n-2}-Ca^{n-3}-\ldots\ldots-Ta.$$

Substituting this value of V in the given equation, and arranging the terms according to the same powers of x and a, we have

$$(x^n-a^n)+A(x^{n-1}-a^{n-1})+B(x^{n-2}-a^{n-2})+\ldots\ldots+T(x-a)=0.$$

As (Art. 83) each of the expressions (x^n-a^n), $(x^{n-1}-a^{n-1})$, etc., is divisible by $x-a$, the given equation is divisible by $x-a$.

Corollary.—Conversely, if the equation

$x^n+Ax^{n-1}+Bx^{n-2}+\ldots+Tx+V=0$, ($n$) is divisible by $x-a$, then a is a root of the equation.

For if the equation (n) is divisible by $x-a$, if we call the quotient Q, we have $(x-a)Q=0$ (n), which may be satisfied by making $x-a=0$, whence $x=a$.

D'ALEMBERT'S PROOF OF PROP. I.—If said division leave a remainder, call it R, and the quotient Q; then, equation (n) becomes

$$(x-a)Q+R=0.$$

But $x-a=0$, $\therefore$ $R=0$; that is, there is no remainder on dividing equation (n) by $x-a$.

ILLUSTRATION.—In the equation $x^3+x^2-14x-24=0$, the roots are -2, -3, and 4; and the equation is divisible by $x+2$, $x+3$, and $x-4$.

396. Proposition II.—*An equation of the* nth *degree has* n *roots.*

Let a be a root of the equation

$$x^n+Ax^{n-1}+Bx^{n-2}+Cx^{n-3}+\ldots+Tx+V=0 \quad (n).$$

By Art. 395 this equation is divisible by $x-a$. If we perform the division, and denote by A_1, B_1, etc., the coëfficients of the powers of x in the quotient, equation (n) becomes

$$(x-a)(x^{n-1}+A_1x^{n-2}+B_1x^{n-3}+\ldots+T_1x+V_1)=0.$$

Hence, $x^{n-1}+A_1x^{n-2}+B_1x^{n-3}+\ldots+T_1x+V_1=0.$

Now, this equation must also have a root, which may be denoted by b, and is (Art. 395) divisible by $x-b$. Hence,

$$(x-b)(x^{n-2}+A_2x^{n-3}+B_2x^{n-4}+\ldots+T_2x+V_2)=0.$$

Placing the second factor of this equation equal to zero, taking c, a third root, and dividing by $x-c$, we shall have an equation of a degree still lower by a unit.

It is evident that if this operation be continued, the exponent n will be exhausted, and the last quotient will be unity; hence, calling the last root l, we shall have

$(x-a)(x-b)(x-c)(x-d)$, . . . $(x-l)=0$, which is satisfied by making $x=a$, b, c, d, ... or l; that is, the equation has n roots, a, b, c, d, etc.

Corollary I.—If we know one root of an equation, by dividing (Art. 395) we may find the equation containing the remaining roots.

Thus, one root of the equation $x^3-12x^2+47x-60=0$, is 5, and by dividing it by $x-5$, the quotient is $x^2-7x+12=0$, the roots of which may be found, viz., $+3$ and $+4$.

Corollary II.—When any equation, whose right hand member is zero, can be separated into factors, the roots of the equation may be found by placing each of the factors equal to zero.

Thus, if $x^2+4x=0$, we have $x(x+4)=0$, whence $x=0$, and $x=-4$. (See Art. 253.)

1. One root of the equation $x^3-11x^2+23x+35=0$ is -1; find the equation containing the remaining roots.

Ans. $x^2-12x+35=0$.

2. One root of the equation $x^3-9x^2+26x-24=0$ is 3; find the remaining roots. Ans. 2 and 4.

3. Two roots of the equation $x^4+2x^3-41x^2-42x+360=0$, are 3 and -4; required the remaining roots.

Ans. 5 and -6.

REMARK.—Two or more of the n roots may be equal to each other. Thus, the equation $x^3-6x^2+12x-8=0$, is the same as $(x-2)(x-2)(x-2)=0$, or $(x-2)^3=0$. Hence, the three roots are $x=2$, $x=2$, and $x=2$.

397. Proposition III.—*No equation of the* n*th* *degree can have more than* n *roots.*

If it be possible let the equation

$$x^n+Ax^{n-1}+Bx^{n-2}+Cx^{n-3}+\ldots\ldots+Tx+V=0,$$

besides the n roots, a, b, c, d, etc., have another root, r, not identical with either of the roots a, b, c, d, etc.; then, the equation must be divisible by $x-r$ (Art. 395); this gives

$$x^n+Ax^{n-1}+Bx^{n-2}+, \text{ etc.}, =(x-r)(x^{n-1}+A'x^{n-2}+, \text{ etc.},) \text{ or}$$
$$(x-a)(x-b)(x-c)\ldots(x-l)=(x-r)(x^{n-1}+A'x^{n-2}+, \text{ etc.})$$

But since r is a value of x, we have, by substitution,

$$(r-a)(r-b)(r-c)\ldots(r-l)=(r-r)(x^{n-1}+A'x^{n-2}+, \text{ etc.})$$

Now, the second member of this equation is $=0$, because $(r-r)=0$; but the other side can not be 0, since r is not equal to any of the quantities a, b, c, etc.; hence, the supposition is absurd that x can have any value other than a, b, c, d, . . l.

398. Proposition IV.—*To discover the relations between the coëfficients of an equation and its roots.*

$$\left.\begin{array}{l}\text{Let } x=a,\\ \quad\;\; x=b,\\ \quad\;\; x=c,\\ \quad\;\; x=d, \text{ etc.}\end{array}\right\}\qquad \text{Then, }\left\{\begin{array}{l}x-a=0,\\ x-b=0,\\ x-c=0,\\ x-d=0, \text{ etc.}\end{array}\right.$$

By multiplying together the corresponding terms of the last set of equations, we have $(x-a)(x-b)(x-c)(x-d)=0$.

If we perform the actual multiplication of the factors, we find

$$\left.\begin{array}{r|r|r|l} x^4-a & x^3+ab & x^2-abc & x+abcd \\ -b & +ac & -abd & \\ -c & +ad & -acd & \\ -d & +bc & -bcd & \\ & +bd & & \\ & +cd & & \end{array}\right\}=0.$$

Similarly, in the equation of the n^{th} degree,

$$x^n+Ax^{n-1}+Bx^{n-2}+, \text{ etc.}, =(x-a)(x-b)(x-c)\ldots(x-l)=0.$$

If we perform the multiplication of the n factors, we shall have $-a-b-c$, etc., $=A$; $ab+ac+ad$, etc., $=B$; $-abc-abd-acd$, etc., $=C$; and so on. For the last term $\pm abcd\ldots kl=V$.

The $\pm$ is prefixed to the last, or *absolute* term, because the product $-a\times-b\times-c\ldots\times-l$, will be *plus* or *minus*, according as the degree of the equation is *even* or *odd*. Hence,

1. *The coëfficient of the second term of any equation is equal to the sum of all the roots, with their signs changed.*

2. *The coëfficient of the third term is equal to the sum of the products of all the roots taken two and two.*

3. *The coëfficient of the fourth term is equal to the sum of the products of all the roots taken three and three, with their signs changed.* And so on; and

4. *The last term is the product of all the roots, with the sign changed if the degree of the equation is odd.*

Corollary I.—*If any term is wanting, its coëfficient is* 0.

II. *If the* 2^d^ *term is wanting, the sum of the roots is* 0.

III. *If the* 3^d^ *term is wanting, the sum of the products of the roots, taken two and two in a product, is* 0.

IV. *If the absolute term is wanting, the product of the roots must be* 0, *and hence one of the roots must be* 0.

V. *Since the last term is the product of all the roots, therefore, it must be divisible by each of them; that is, every rational root of an equation is a divisor of the last term.*

EXAMPLES ILLUSTRATING THE PRECEDING PRINCIPLES.

1. Form the equation whose roots are 3, 4, and —5.

The equations $x=3$, $x=4$, and $x=-5$, give $x-3=0$, $x-4=0$, and $x+5=0$; hence, $(x-3)(x-4)(x+5)=x^3-2x^2-23x+60=0$. Here, $3+4-5=+2$, coëfficient of 2^d term with contrary sign.

$3\times4+3\times-5+4\times-5=-23$, the coëfficient of the 3^d term.

$3\times4\times-5=-60$, the sign of which must be changed, for the last term, because the degree of the equation is odd.

2. What is the equation whose roots are 2, 3, and —5? (See Cor. 2.) Ans. $x^3-19x+30=0$.

3. Form the equation with roots 0, —1, 2, and —5. Ans. $x^4+4x^3-7x^2-10x=0$.

4. Find the equation whose roots are $1\pm\sqrt{2}$ and $2\pm\sqrt{3}$. Ans. $x^4-6x^3+8x^2+2x-1=0$.

5. What is the 4^{th} term of the equation whose roots are $-2, -1, 1, 3, 4$? Ans. $29x^2$.

399. Proposition V.—*No equation having unity for the coëfficient of the first term, and all the other coëfficients integers, can have a root equal to a rational fraction.*

Assume that all the coëfficients are integers in the general equation,

$$x^n+Ax^{n-1}+Bx^{n-2}+\ldots\ldots+Tx+V=0.$$

If possible, let $\frac{a}{b}$, a fraction in its lowest terms, be a root of this equation; then, by substituting it for x, reducing the terms to a common denominator, transposing, etc., we shall have

$$\frac{a^n}{b}=-Aa^{n-1}-Ba^{n-2}b-\ldots\ldots-Tab^{n-2}-Vb^{n-1}.$$

But, by hypothesis, a and b, and, consequently, a^n and b, contain no common factor; therefore, an irreducible fraction is equal to a series of integers, which is *absurd*. Hence, the supposition is absurd, and the equation has no fractional root.

400. Proposition VI.—*If the signs of the alternate terms of an equation be changed, the signs of all the roots will be changed.*

Let a be a root of the equation

$$x^n+Ax^{n-1}+Bx^{n-2}+Cx^{n-3}+\ldots+V=0 \quad (1);$$

Then, $a^n+Aa^{n-1}+Ba^{n-2}+Ca^{n-3}+\ldots+V=0 \quad (2)$.

Changing the signs of the alternate terms of equation (1),

$$x^n-Ax^{n-1}+Bx^{n-2}-Cx^{n-3}+\ldots\pm V=0 \quad (3).$$

Substituting $-a$ for x in this equation, we have

$$a^n-Aa^{n-1}+Ba^{n-2}-Ca^{n-3}\ldots\pm V=0 \quad (4).$$

Now, if n be *even*, the 2^d, 4^{th}, etc., terms will contain odd powers of $-a$, which will (Art. 193) render those terms positive. Hence, the whole result will be the same as that produced by the substitution of a for x in equation (1).

But if n be *odd*, the 1st, 3d, etc., terms will be negative, which will render all the terms of (4) negative. Changing the signs of all the terms, (4) becomes the same as (2).

Hence, if a is a root of equation (1), $-a$ is a root of (3), whether n be odd or even.

Ex.—The roots of the equation $x^3-3x^2-10x+24=0$, are 2, -3, and 4; what are the roots of the equation $x^3+3x^2-10x-24=0$? Ans. -2, 3, and -4.

401. Proposition VII.—*When the coëfficients of an equation are real, if it contains imaginary roots, the number of these roots must be even.*

If $a+b\sqrt{-1}$ be a root of the eq. $x^n-Ax^{n-1}+Bx^{n-2}-Cx^{n-3}+$, etc., $=0$; then, $a-b\sqrt{-1}$ is also a root.

In the equation, substitute $a+b\sqrt{-1}$ for x, and the result will consist of two parts:

1st. Possible quantities which involve the odd and even powers of a, and the *even* powers of $b\sqrt{-1}$;

2d. Impossible quantities which involve the *odd* powers of $b\sqrt{-1}$.

Call the sum of the possible quantities P, and of the impossible $Q\sqrt{-1}$; then, $P+Q\sqrt{-1}$ is the whole result; hence, $P+Q\sqrt{-1}=0$.

But the first quantity being real, and the second imaginary, in order to satisfy the equation, each of the quantities must be 0; this gives $P=0$, and $Q\sqrt{-1}=0$.

Again, let $a-b\sqrt{-1}$ be substituted for x, and the 1st part of the result will be the same as before, and the 2d part, which arises from the odd powers of $b\sqrt{-1}$, will differ from the former imaginary part only in its sign; therefore, the result will be $P-Q\sqrt{-1}$; but since $P=0$, and $Q\sqrt{-1}=0$, we must have $P-Q\sqrt{-1}=0$.

Hence, $a-b\sqrt{-1}$ is a root of the equation, since its substitution for x gives a result equal to 0.

Corollary I.—In a similar manner, it may be shown that surd roots of the form $a\pm\sqrt{b}$, $\pm b\sqrt{-1}$, or $\pm\sqrt{b}$, enter an equation by pairs.

Corollary II.—Since irrational and imaginary roots always occur in pairs where the coëfficients are real, it fol-

lows that every equation of an odd degree must have at least *one* real root.

Corollary III.—Corresponding to any pair of imaginary roots $a\pm b\sqrt{-1}$, we have in the eq. the quadratic factor,

$$\{x-(a+b\sqrt{-1})\}\{x-(a-b\sqrt{-1})\}=(x-a)^2+b^2;$$

Hence, every eq. of an *even* order, with real coëfficients, is composed of *real* factors of the second degree.

1. One root of the equation $x^3-26x+60=0$ is -6; required the other roots. Ans. $3\pm\sqrt{-1}$.

2. One root of $x^3-7x^2+13x-3=0$, is $2-\sqrt{3}$; find the other roots. Ans. $2+\sqrt{3}$ and 3.

3. One root of $x^4-3x^2-42x-40=0$ is $-\frac{1}{2}(3+\sqrt{-31})$; find the other roots. Ans. $-\frac{1}{2}(3-\sqrt{-31})$, 4, and -1.

4. Two roots of $x^5-10x^4+29x^3-10x^2-62x+60=0$ are 3 and $\sqrt{2}$; find the other roots. A. $-\sqrt{2}$, 2, and 5.

402. Proposition VIII.—Descartes' Rule of the Signs.—*No equation can have a greater number of* POSITIVE *roots than there are* VARIATIONS *of sign; nor a greater number of* NEGATIVE *roots than there are* PERMANENCES *of sign.*

In the equation $x-a=0$, where the value of x is $+a$, there is *one variation*, and *one positive root.*

In the equation $x+a=0$, where the value of x is $-a$, there is *one permanence*, and *one negative root.*

In $x^2-(a+b)x+ab=0$, where the values of x are $+a$ and $+b$, there are *two variations* and *two positive* roots.

In $x^2+(a+b)x+ab=0$, where the values of x are $-a$, and $-b$, there are *two permanences*, and *two negative* roots.

In $x^2-x-12=0$, where $x=+4$, and -3, there is *one variation*, and *one positive* root, *one permanence*, and *one negative* root.

If we form an equation of the third degree, (Art. 397), whose roots are $+2$, $+3$, $+4$, we shall have $x^3-9x^2+26x-24=0$, where there are *three variations*, and *three positive* roots.

But if we form an equation whose roots are -2, -3, $+4$, we shall have $x^3+x^2-14x-24=0$, where there is *one variation*, and *one positive* root, and *two permanences*, and *two negative* roots.

To prove the proposition generally, let the signs of the terms in their order, in any *complete* equation be

$+ + - + - + + +$, and let a new factor $x-a=0$, corresponding to a new positive root be introduced, the signs in the partial and final products will be

$$\begin{array}{ccccccccc}
+ & + & - & + & - & + & + & + & \\
+ & - & & & & & & & \\
\hline
+ & + & - & + & - & + & + & + & \\
 & - & - & + & - & + & - & - & - \\
\hline
+ & \pm & - & + & - & + & \pm & \pm & -.
\end{array}$$

Now, in this product, it is obvious, that *each permanence is changed into an ambiguity;* hence, the permanences, take the ambiguous sign as you will, are not *increased* in the final product; but the number of signs is increased by *one*, and therefore the number of variations must be increased by *one*.

Hence, the introduction of any positive root introduces at least one additional variation of sign.

Let us now begin with the equation $x-a=0$, which contains one positive root, and has one variation of sign. Then, since every additional positive root introduces at least one additional variation of sign, *the number of positive roots can never exceed the number of variations of sign.*

Again, if we change the signs of the alternate terms, the roots will be changed from positive to negative, and, conversely, (Art. 400), the permanences and variations, in the proposed equation, will be interchanged.

But since the changed equation can not have a greater number of positive roots than there are variations of sign, *the proposed equation can not have a greater number of negative roots than there are permanences of sign.*

Corollary I.—In an equation of the m^{th} degree, since the sum of the variations and permanences is equal to m, the number of *real* roots in any equation can not be greater than its degree.

Corollary II.—If the number of *real* roots be less than the degree of the equation, the remaining roots are *imaginary.*

Take, for example, the equation

$$x^2+16=0, \text{ or } x^2\pm 0x+16=0.$$

Taking the upper sign, there are no variations; hence, there is no positive root: taking the lower sign, there are no permanences; hence, there is no negative root. But the equation has two roots (Art. 396); they must, therefore, both be imaginary.

Take, again, the cubic equation

$$x^3+Bx+C=0, \text{ or } x^3\pm 0x^2+Bx+C=0.$$

Reasoning as before, we find that there can be but one *real* root, which is negative. Therefore, the other two roots must be imaginary.

403. Proposition IX.—*If two numbers, when substituted for the unknown quantity in an equation, give results affected with different signs, one root, at least, of this equation lies between these numbers.*

Let the equation, for example, be $x^3-x^2+x-8=0$.

If we substitute 2 for x in this equation, the result is -2; and if we substitute 3 for x, the result is $+13$. It is required to show that there must be one real root, at least, between 2 and 3.

The equation may evidently be written thus,

$$(x^3+x)-(x^2+8)=0.$$

Now, in substituting 2 for x, $x^3+x=10$, and $x^2+8=12$;

$$\text{Therefore, } x^3+x<x^2+8;$$

Also, in substituting 3 for x, $x^3+x=30$, and $x^2+8=17$;

$$\text{Therefore, } x^3+x>x^2+8.$$

Now, both members of the inequality increase while x increases, but the first increases more rapidly than the second, since when $x=2$, it is *less* than the second, but when $x=3$, it is *greater*. Consequently, for some value of x between 2 and 3, we must have $x^3+x=x^2+8$, and this value of x is, therefore, a real root.

In general, suppose $X=0$ to be a polynomial equation involving x, and that p and q, when substituted for x, give results with contrary signs. Let P be the sum of the positive, and N the sum of the negative terms. When $x=p$, let P—N be negative, or $P<N$; and when $x=q$, let P—N be positive, or $P>N$.

Now, there must be some value of x between p and q, which renders $P=N$, or satisfies the equation $X=0$. This value of x is, therefore, a real root of the equation.

Corollary.—If the difference between p and q is equal to *unity*, it is evident that we have found the *integral* part of one of the roots.

1. Find the integral part of one value of x in the equation

$$x^4-4x^3+3x^2+x-5=0.$$

If $x=3$, the value of the expression is -2; but if $x=4$, the value is 47. Hence, 3 is the first figure of one root.

2. Required the first figure of one of the roots of the equation $x^3-5x^2-x+1=0$. Ans. 5.

TRANSFORMATION OF EQUATIONS.

404. The **Transformation of an Equation** is the changing of it into another of the same degree, whose roots shall have a specified relation to the roots of the given equation.

Thus, in $x^n+Ax^{n-1}+Bx^{n-2}. \quad . \quad . +Tx+V=0$; (1)

if $-y$ be substituted for x, the equation will be transformed into another whose roots are the same as those in (1), but with contrary signs, for $y=-x$.

If $\frac{1}{y}$ be substituted for x; then, $y=\frac{1}{x}$, and the roots of the new equation in y will be the reciprocals of those of equation (1).

405. Proposition I.—*To transform an equation into one whose roots are the roots of the given equation multiplied or divided by any given quantity.*

Let a, b, c, etc., be the roots of the equation

$$x^n+Ax^{n-1}+Bx^{n-2}. \quad . \quad . \quad . \quad +Tx+V=0. \quad (1).$$

Assume $y=kx$, or $x=\frac{y}{k}$. Substituting this value for x, in (1),

$$\frac{y^n}{k^n}+A\frac{y^{n-1}}{k^{n-1}}+B\frac{y^{n-2}}{k^{n-2}}. \quad . \quad . \quad \quad . \quad . \quad +\frac{Ty}{k}+V=0;$$

Hence, $y^n+Aky^{n-1}+Bk^2y^{n-2}. \quad . \quad . \quad . \quad +Tk^{n-1}y+k^nV=0.$

Since $y=kx$, the roots of this eqnation are ka, kb, kc, etc.

It is evident that this equation may be derived from (1); or that the transformation of (1) is effected, by multiplying the successive terms by 1, k, k^2, k^3, etc., and changing x into y.

In the case of division, assume $y=\frac{x}{k}$, or $x=ky$, and substitute.

Corollary.—By this transformation an equation may be cleared of fractions, or the coëfficient of the first term may be made *unity*.

1. Let it be required to transform the equation

$$x^3+\tfrac{1}{2}px^2+\tfrac{1}{3}qx+r=0,$$

into one which is clear of fractions, and which has unity for the coëfficient of the term containing the highest power of x.

Multiplying by 6, $6x^3+3px^2+2qx+6r=0$.

Putting $y=6x$, or $x=\frac{1}{6}y$, $6\frac{y^3}{6^3}+3p\frac{y^2}{6^2}+2q\frac{y}{6}+6r=0$;

Multiplying by 6^2, $y^3+3py^2+12qy+216r=0$.

2. Find the equation whose roots are each 3 times those of the equation $x^4+7x^2-4x+3=0$.

Ans. $y^4+63y^2-108y+243=0$.

3. Find the equation whose roots are each 5 times those of the equation $x^4+2x^3-7x-1=0$.

Ans. $y^4+10y^3-875y-625=0$.

4. What is the equation whose roots are each $\frac{1}{2}$ of those of $x^3-3x^2+4x+10=0$? Ans. $4y^3-6y^2+4y+5=0$.

5. Transform eq. $x^3-2x^2+\frac{1}{3}x-10=0$, into one having integral coëfficients. Ans. $y^3-6y^2+3y-270=0$.

406. Proposition II.—*To transform an equation into one whose roots are greater or less by any given quantity than the corresponding roots of the proposed equation.*

Let $x^n+Ax^{n-1}+Bx^{n-2} \ldots \ldots +Tx+V=0$, be an equation whose roots are a, b, c, etc.

The relation between x and y will be expressed by the equation $y=x\pm r$. As the principle is the same in both cases, let $y=x-r$, or $x=y+r$. Substituting $y+r$ for x, we have

$$(y+r)^n+A(y+r)^{n-1}+B(y+r)^{n-2} \qquad +T(y+r)+V=0.$$

Developing the different powers of $y+r$ by the Binomial Theorem, and arranging the terms, we have

$$\left.\begin{array}{r|r|l}
y^n+nr & y^{n-1}+\dfrac{n(n-1)}{1\cdot 2}r^2 & y^{n-2}. \quad . \quad . \quad . \qquad +r^n \\
+A & +(n-1)Ar & \qquad\qquad\qquad\quad +Ar^{n-1} \\
 & +B & \qquad\qquad\qquad\quad +Br^{n-2} \\
 & & \qquad\qquad\qquad\quad \vdots \\
 & & \qquad\qquad\qquad\quad +Tr \\
 & & \qquad\qquad\qquad\quad +V
\end{array}\right\}=0.$$

Now, since $y=x-r$, the values of y in this equation are $a-r$, $b-r$, $c-r$, etc.

407. Corollary.—By means of the preceding transformation, we may remove any intermediate term of an equation. Thus, to transform an equation into one which shall want the second term, r must be assumed so that $nr+A=0$.

To take away the third term, put $\frac{1}{2}n(n-1)r^2+(n-1)Ar+B=0$.

1. Transform the equation $x^3-7x+7=0$ into another whose roots shall be less by one than the corresponding roots of this equation. Ans. $y^3+3y^2-4y+1=0$.

2. Find the equation whose roots are less by 3 than those of the equation $x^4-3x^3-15x^2+49x-12=0$.
Ans. $y^4+9y^3+12y^2-14y=0$.

3. Transform eq. $x^3-6x^2+8x-2=0$ into another whose second term shall be absent. Ans. $y^3-4y-2=0$.

408. There is an easier and more elegant method of transformation, which we will now proceed to explain.

Let the proposed equation be

$$Ax^4+Bx^3+Cx^2+Dx+E=0, \qquad (1)$$

and let it be required to transform it into another, whose roots shall be less by r; then, $y=x-r$ and $x=y+r$.

By substituting $y+r$, instead of x, we have

$$A(y+r)^4+B(y+r)^3+C(y+r)^2+D(y+r)+E=0.$$

By developing the powers of $y+r$, and arranging, as in Art. 406, the transformed equation will take the form

$$Ay^4+B_1y^3+C_1y^2+D_1y+E_1=0, \qquad (2)$$

where A is evidently the same as in (1), while B_1, C_1, D_1, and E_1, are unknown quantities to be determined. For y, substitute its value $x-r$, and equation (2) becomes

$$A(x-r)^4+B_1(x-r)^3+C_1(x-r)^2+D_1(x-r)+E_1=0. \quad (3)$$

Now, since the values of x are the same in (1) and (3), these equations are identical. Hence, any operation may be performed on (1) or (3) with the same result.

Now, as the object is to obtain the values of B_1, C_1, etc., let (3) or (1) be divided by $x-r$, and the quotient will be

$$A(x-r)^3+B_1(x-r)^2+C_1(x-r)+D_1,$$

with the remainder E_1; hence, E_1 is determined.

Divide this quotient by $x-r$, and the next quotient will be

$$A(x-r)^2+B_1(x-r)+C_1,$$

with a remainder D_1; hence, D_1 is determined.

Continuing the division by $x-r$, we obtain C_1 and B_1, and thus find all the coëfficients of equation (2).

To illustrate, let us now solve Ex. 1, Art. 407, by this method.

Transform the equation $x^3-7x+7=0$ into another, whose roots shall be less by 1 than the corresponding roots of this equation.

Here, $y=x-1$, and we proceed to divide the proposed equation and the successive quotients by $x-1$. The successive remainders will be the coëfficients of y in the transformed equation, except that of the highest power, which will have the same coëfficient as the highest power of x in the proposed equation.

$$\begin{array}{lr}
x-1)x^3-7x+7(x^2+x-6 & \\
\quad x^3-x^2 & \text{1st quot.} \\
\hline
\quad +x^2-7x & \\
\quad\ \ x^2-\ x & \\
\hline
\qquad -6x+7 & \\
\qquad -6x+6 & \\
\hline
\text{1st rem.}\quad =+1 &
\end{array}
\qquad
\begin{array}{lr}
x-1)x^2+x-6(x+2 & \\
\quad x^2-x & \text{2d quot.} \\
\hline
\quad +2x-6 & \\
\quad\ \ 2x-2 & \\
\hline
\text{2d rem.}\quad =-4 & \\
 & \\
x-1)x+2(1, & \text{3d quot.} \\
\quad x-1 & \\
\hline
\text{3d rem.}\quad =+3 &
\end{array}$$

Since the successive remainders are $+3$, -4, and $+1$, we have $A=1$, $B_1=+3$, $C_1=-4$, and $D_1=+1$. Hence, the transformed equation is $y^3+3y^2-4y+1=0$.

This method of transforming an equation may be greatly shortened by *Horner's Synthetic Method of Division*, which we shall now proceed to explain.

409. Synthetic Division.—This may be considered as an abridgment of the method of division by Detached Coëfficients (Art. 77). To explain the process, we shall first divide $5x^4-12x^3+3x^2+4x-5$ by $x-2$, by detached coëfficients.

By changing the sign of the second term of the divisor, and *adding* each partial product, except the *first* term, which, being always the same as the first term of each dividend, may be *omitted*, the operation may be represented as in the margin below:

$$\begin{array}{l}
\text{Divisor.} \qquad\qquad\qquad\qquad \text{Quotient.}\\
1-2)5-12+3+4-5(5-2-1+2,\\
\quad\ \ \underline{5-10} \qquad\quad \text{or } 5x^3-2x^2-x+2\\
\qquad\ \ -2+3\\
\qquad\ \ \underline{-2+4}\\
\qquad\qquad -1+4\\
\qquad\qquad \underline{-1+2}\\
\qquad\qquad\quad +2-5\\
\qquad\qquad\qquad \underline{2-4}\\
\qquad\qquad\qquad -1 \text{ R}
\end{array}$$

Let it be observed that the figures over the stars are the coëfficients of the several terms of the quotient; also, that it is unnecessary to bring down the several terms of the dividend.

Hence, the last operation may be represented as follows:

$$\begin{array}{l}
1+2)5-12+3+4-5(5-2-1+2\\
\quad\ \ \underline{*+10}\\
\qquad -2+3\\
\qquad\ \ \underline{*-4}\\
\qquad\qquad -1+4\\
\qquad\qquad\ \ \underline{*-2}\\
\qquad\qquad\quad +2-5\\
\qquad\qquad\qquad \underline{*+4}\\
\qquad\qquad\qquad\ \ -1
\end{array}$$

$$\begin{array}{r}
+2)5-12+3+4-5\\
\underline{+10-4-2+4}\\
-\ 2-1+2-1
\end{array}$$

In this operation, 5 is the first term of the quotient, $+10$ is the product of $+2$, the divisor, by 5; the sum of $+10$ and -12 gives -2, the second term of the quotient; $+2\times-2=-4$, and -4 and $+3$ gives -1, the third term of the quotient, and so on. The last term, -1, is the remainder.

Supplying the powers of x, the quotient is $5x^3-2x^2-x+2$, with a remainder -1.

A similar method may be used when the divisor contains three terms, but the process is more complicated.

If the coëfficient of the first term of the divisor is not unity, it may be made unity by dividing both dividend and divisor by the coëfficient of the first term of the divisor.

If any term is wanting, its place must be supplied with a zero.

410. In application of these principles,

1. Let it be required to find the equation whose roots are less by 1 than those of the equation x^3-7x+7.

Since the second term is wanting, its place must be supplied with 0. The divisor is $x-1$; hence, we divide by $+1$.

OPERATION BY SYNTHETIC DIVISION.

$$\begin{array}{llllll}
+1) & 1 & \pm0 & -7 & +7 & \\
 & & +1 & +1 & -6 & \\
\hline
 & & +1 & -6 & +1 & \therefore +1= 1^{st} \text{ R.} \\
 & & +1 & +2 & & \\
\hline
 & & +2 & -4 & \therefore -4= 2^{d} \text{ R.} & \\
 & & +1 & & & \\
\hline
 & & +3 & \therefore +3= 3^{d} \text{ R.} & &
\end{array}$$

Hence, the required coëfficients are 1, $+3$, -4, and $+1$.

$\therefore y^3+3y^2-4y+1=0$ is the transformed equation required.

2. Transform the equation $5x^4+28x^3+51x^2+32x-1=0$, into another having its roots greater by 2 than those of the given equation.

Here, $y=x+2$; hence, we divide by -2, thus,

$$\begin{array}{lllllll}
-2) & 5 & +28 & +51 & +32 & -1 & \\
 & & -10 & -36 & -30 & -4 & \\
\hline
 & & +18 & +15 & +2 & -5 & \therefore -5= 1^{st} \text{ R.} \\
 & & -10 & -16 & +2 & & \\
\hline
 & & +8 & -1 & +4 & \therefore +4= 2^{d} \text{ R.} & \\
 & & -10 & +4 & & & \\
\hline
 & & -2 & +3 & \therefore +3= 3^{d} \text{ R.} & & \\
 & & -10 & & & & \\
\hline
 & & -12 & \therefore -12= 4^{th} \text{ R.} & & &
\end{array}$$

Hence, $A=5$, $B_1=-12$, $C_1=+3$, $D_1=+4$, and $E_1=-5$ $\therefore$ the transformed equation is $5y^4-12y^3+3y^2+4y-5=0$.

3. Find the equation whose roots are less by 1.7 than those of the equation $x^3-2x^2+3x-4=0$.

If we transform this equation into another whose roots are less by 1, the resulting equation is $y^3+y^2+2y-2=0$. We may then transform this into another whose roots are less by .7, or the whole operation may be performed at once, as follows:

$$
\begin{array}{lllll}
+1.7) & 1 & -2 & +3 & -4 \\
 & & +1.7 & .51 & +4.233 \\
\hline
 & & -\ .3 & +2.49 & +\ .233\ ..\ +.233 = 1^{st}\ \text{R.} \\
 & & +1.7 & +2.38 & \\
\hline
 & & +1.4 & +4.87\ \therefore\ +4.87 = 2^{d}\ \text{R.} & \\
 & & +1.7 & & \\
\hline
 & & +3.1\ \therefore\ 3.1 = 3^{d}\ \text{R.} & &
\end{array}
$$

Hence, the equation is $y^3+3.1y^2+4.87y+.233=0$.

4. Find the equation whose roots are each less by 3 than the roots of $x^3-27x-36=0$. Ans. $y^3+9y^2-90=0$.

5. Required the equation whose roots are less by 5 than those of the equation $x^4-18x^3-32x^2+17x+9=0$.

Ans. $y^4+2y^3-152y^2-1153y-2331=0$.

6. Required the equation whose roots are less by 1.2 than those of the equation $x^5-6x^4+7.4x^3+7.92x^2-17.872x-.79232=0$. Ans. $y^5-7y^3+2y-8=0$.

Transform the following equations into others wanting the 2d term. (See Art. 407.)

7. $x^3-6x^2+7x-2=0$. Ans. $y^3-5y-4=0$.

8. $x^3-6x^2+12x+19=0$. Ans. $y^3+27=0$.

Transform the following equations into others wanting the 3d term:

9. $x^3-6x^2+9x-20=0$.

Ans. $y^3+3y^2-20=0$, or $y^3-3y^2-16=0$.

10. $x^3-4x^2+5x-2=0$.

Ans. $y^3-y^2=0$, or $y^3+y^2-\frac{4}{27}=0$.

411. Proposition III.—*To determine the law of Derived Polynomials.*

Let X represent the general equation of the n^{th} degree; that is,

$$X = x^n + Ax^{n-1} + Bx^{n-2} \ldots + Tx + V = 0.$$

If we substitute $x+h$ for x, and put X_1 to represent the new value of X, we have

$$X_1 = (x+h)^n + A(x+h)^{n-1} + B(x+h)^{n-2} +, \text{ etc.},$$

and if we expand the different powers of $x+h$ by the binomial theorem, we have $X_1 =$

$$\begin{array}{l|l|l|l}
x^n & + \quad nx^{n-1} & h + \quad n(n-1)x^{n-2} & \frac{h^2}{1\cdot 2} +, \text{ etc.} \\
+Ax^{n-1} & +(n-1)Ax^{n-2} & +(n-1)(n-2)Ax^{n-3} & \\
+Bx^{n-2} & +(n-2)Bx^{n-3} & +(n-2)(n-3)Bx^{n-4} & \\
+, \text{ etc.} & +, \text{ etc.} & +, \text{ etc.} &
\end{array}$$

But the first vertical column is the same as the original equation, and if we put X′, X″, X‴, etc., to represent the succeeding columns, we have

$$\begin{aligned}
X\ &= x^n + Ax^{n-1} + Bx^{n-2} +, \text{ etc.},\\
X' &= nx^{n-1} + (n-1)Ax^{n-2} + (n-2)Bx^{n-3} +, \text{ etc.},\\
X'' &= n(n-1)x^{n-2} + (n-1)(n-2)Ax^{n-3} +, \text{ etc.},\\
&\text{Etc.,} \qquad\qquad \text{etc.}
\end{aligned}$$

By substituting these in the development of X_1, we have

$$X_1 = X + X'h + \frac{X''}{1\cdot 2}h^2 + \frac{X'''}{1\cdot 2\cdot 3}h^3 +, \text{ etc.}$$

The expressions X′, X″, X‴, etc., are called *derived polynomials* of X, or *derived functions* of X. X′ is called the *first derived polynomial* of X, or *first derived function* of X; X″ is called the *second*, X‴ the *third*, and so on.

It is easily seen that X′ may be derived from X, X″ from X′, etc., *by multiplying each term by the exponent of x in that term, and diminishing the exponent by unity.*

412. Corollary.—If we transpose X, we have $X_1-X=X'h+\frac{X''}{1\cdot 2}h^2+$, etc. Now, it is evident that h may be taken so small that *the sign of the sum* $X'h+\frac{X''}{1\cdot 2}h^2+$, etc., *will be the same as the sign of the first term* $X'h$.

For, since $X'h+\frac{1}{2}X''h^2+$, etc., $=h(X'+\frac{1}{2}X''h+$, etc.), if h be taken so small, that $\frac{1}{2}X''h+\frac{1}{6}X'''h^2+$, etc., becomes less than X' (their magnitudes alone being considered), the sign of the sum of these two expressions must be the same as the sign of the greater X'.

413. By comparing the transformed equation in Art. 406, with the development of X_1 in Art. 411, it is easily seen that X_1 may be considered the transformed equation, y corresponding to x, and r to h.

Hence, the tranformed equation may be obtained by substituting the values of X, X′, etc., in the development of X_1. As an example,

Let it be required to find the equation whose roots are less by 1 than those of the equation $x^3-7x+7=0$.

Here, . . . $X=x^3-7x+7$, $X'''=6$,
$X'=3x^2-7$, $X^{iv}=0$.
$X''=6x$,

Observing that $h=1$, and substituting these values in the equation $X_1=X+X'h+\frac{X''}{1\cdot 2}h^2+\frac{X'''}{1\cdot 2\cdot 3}h^3+$, etc., we have $X_1=(x^3-7x+7)+(3x^2-7)1+(6x)\frac{1}{1\cdot 2}+\frac{6}{1\cdot 2\cdot 3}=x^3+3x^2-4x+1$, in which the value of x is equal to that of x in the given equation diminished by 1.

By this method, solve the examples in Art. 410.

EQUAL ROOTS.

414. *To determine the equal roots of an equation.*

We have already seen (Art. 396, Rem.) that an equation may have two or more of its roots equal to each other. We now propose to determine when an equation has equal roots, and how to find them.

If we take the equation $(x-2)^3=0$ (1), its first derived polynomial is $3(x-2)^2=0$.

Hence, we see that if any equation contains the same factor taken *three* times, its first derived polynomial will contain the same factor taken *twice;* this last factor is, therefore, *a common divisor* of the given equation, and its first derived polynomial.

In general, if we have an equation $X=0$, containing the factors $(x-a)^m(x-b)^n$, its first derived polynomial will contain the factors $m(x-a)^{m-1}n(x-b)^{n-1}$; that is, the *greatest common divisor* of the given equation, and its first derived polynomial, will be $(x-a)^{m-1}(x-b)^{n-1}$, and the given equation will have m roots, each equal to a, and n roots, each equal to b

Therefore, to determine whether an equation has equal roots,

Find the greatest common divisor between the equation and its first derived polynomial. If there is no common divisor, the equation has no equal roots.

If the G.C.D. contains a factor of the form $x-a$, then it has *two* roots equal to a; if it contains a factor of the form $(x-a)^2$ it has *three* roots equal to a, and so on.

If it has a factor of the form $(x-a)(x-b)$ it has two roots equal to a, and two equal to b, and so on.

1. Given the equation $x^3-x^2-8x+12=0$, to determine whether it has equal roots, and if so, to find them.

We have for the first derived polynomial (Art. 411), $3x^2-2x-8$.

The G.C.D. of this and the given equation (Art. 108) is $x-2$. Hence, $x-2=0$, and $x=+2$. Therefore, the equation has two roots equal to 2.

Now, since the equation has *two* roots equal to 2, it must be divisible by $(x-2)(x-2)$, or $(x-2)^2$. (Art. 395). Whence,

$$x^3-x^2-8x+12=(x-2)^2(x+3)=0, \text{ and } x+3=0, \text{ or } x=-3.$$

Hence, when an equation contains other roots besides the equal roots, the degree of the equation may be depressed by division, and the unequal roots found by other methods.

The following equations have equal roots; find all the roots.

2. $x^3-2x^2-15x+36=0$. . . . Ans. 3, 3, -4.
3. $x^4-9x^2+4x+12=0$. . . Ans. 2, 2, -1, -3.
4. $x^4-6x^3+12x^2-10x+3=0$. Ans. 1, 1, 1, 3.
5. $x^4-7x^3+9x^2+27x-54=0$. Ans. $x=3, 3, 3, -2$.
6. $x^4+2x^3-3x^2-4x+4=0$. Ans. $-2, -2, +1, +1$.
7. $x^4-12x^3+50x^2-84x+49=0$. A. $3\pm\sqrt{2}$, $3\pm\sqrt{2}$.
8. $x^5-2x^4+3x^3-7x^2+8x-3=0$.
   Ans. 1, 1, 1, $-\frac{1}{2}\pm\frac{1}{2}\sqrt{-11}$.
9. $x^6+3x^5-6x^4-6x^3+9x^2+3x-4=0$.
   Ans. 1, 1, 1, -1, -1, -4.

Suggestion.—In the solution of equations of high degree, the principles above explained may be extended. Thus, in the last example, the G.C.D. is x^3-x^2-x+1. Proceeding, we may, 1st, find the common measure of this and its first derived polynomial, and thus resolve into factors; or, 2d, find the G.C.D. of the first and second derived polynomials. If it is of the form $x-a$, one of the factors of the original equation will evidently be $(x-a)^3$, etc.

By the 1st method, we find $x^3-x^2-x+1=(x-1)(x^2-1)=(x-1)^2(x+1)$; by the 2d, $(x-1)^3$ is a factor of the original equation; hence, $(x-1)^2$ is a factor of x^3-x^2-x+1.

LIMITS OF THE ROOTS OF EQUATIONS.

415. Limits to a Root of an Equation are any two numbers between which that root lies.

A **Superior Limit** to the positive roots is a number numerically greater than the greatest positive root.

Its characteristic is, that when it, or any number greater than it, is substituted for x in the equation, the result is *positive.*

An **Inferior Limit** to the negative roots, is a number numerically greater than the greatest negative root. The substitution of it, or any number greater than it, for x, produces a *negative* result.

The object of ascertaining the limits of the roots is to diminish the labor necessary in finding them.

416. Proposition I.—*The greatest negative coëfficient, increased by unity, is greater than the greatest root of the equation.*

Take the general equation

$$x^n+Ax^{n-1}+Bx^{n-2} \quad \ldots\ldots +Tx+V=0,$$

and suppose A to be the greatest negative coëfficient.

The reasoning will not be affected if we suppose all the coëfficients to be negative, and each equal to A.

It is required to find what number substituted for x will make $x^n>A(x^{n-1}+x^{n-2}+x^{n-3}.\ \ldots\ +x+1)$.

By Art. 297, the sum in parenthesis is $\frac{x^n-1}{x-1}$; hence, we must have $x^n>A\left(\frac{x^n-1}{x-1}\right)$, or $x^n>\frac{Ax^n}{x-1}-\frac{A}{x-1}$.

But if $x^n=\frac{Ax^n}{x-1}$, we find $x=A+1$; $\therefore$ A+1 substituted for x will render $x^n=\frac{Ax^n}{x-1}$, and, consequently, $x^n>\frac{Ax^n}{x-1}-\frac{A}{x-1}$.

By considering all the coëfficients after the first negative, we have taken the most unfavorable case; if any of them, as B, were positive, the quantity in parenthesis would be *less.*

417. Proposition II.—*If we take the greatest negative coëfficient, extract a root of it whose index is equal to the number of terms preceding the first negative term, and increase it by unity, the result will be greater than the greatest positive root of the equation.*

Let Cx^{n-r} be the first negative term, C being the greatest negative coëfficient; then, any value of x which makes

$$x^n > C(x^{n-r}+x^{n-r-1} \cdot \cdot \cdot \cdot \cdot +x+1) \quad (1)$$

will render the first of the proposed equation >0, or positive; because this supposes all the coëfficients after C negative, and each equal to the greatest, which is evidently the most unfavorable case.

By Art. 297, the series in parenthesis $=\dfrac{x^{n-r+1}-1}{x-1}$. Hence,

$$x^n > C\left(\frac{x^{n-r+1}-1}{x-1}\right), \text{ or } x^n > \frac{Cx^{n-r+1}}{x-1} - \frac{C}{x-1}. \text{ But,}$$

this inequality will be true if $x^n = \dfrac{Cx^{n-r+1}}{x-1}$, or $> \dfrac{Cx^{n-r+1}}{x-1}$;

or, by multiplying both members by $x-1$, and dividing by x^{n-r+1}, when $(x-1)x^{r-1}=C$, or $>C$ (2).

But $x-1$ is $<x$, and $\therefore$ $(x-1)^{r-1}<x^{r-1}$ $\therefore$ (2) will be true if we have $(x-1)(x-1)^{r-1}$,

Or $(x-1)^r=C$, or $>C$;

Or $x-1=\sqrt[r]{C}$, or $>\sqrt[r]{C}$;

Or $x=1+\sqrt[r]{C}$, or $>1+\sqrt[r]{C}$.

Find superior limits of the roots of the following equations:

1. $x^4-5x^3+37x^2-3x+39=0$.

Here, $C=5$, and $r=1$ $\therefore$ $1+\sqrt[r]{C}=1+5^{\frac{1}{1}}=6$, Ans.

2. $x^5+7x^4-12x^3-49x^2+52x-13=0$.

Here, $1+\sqrt[r]{C}=1+\sqrt[2]{49}=1+7=8$, Ans.

3. $x^4+11x^2-25x-67=0$.

By supposing the second term $+0x^3$, we have $r=3$; hence, the limit is $1+\sqrt[3]{67}$, or 6.

4. $3x^3-2x^2-11x+4=0$.

Dividing by 3, $x^3-\frac{2}{3}x^2-\frac{11}{3}x+\frac{4}{3}=0$.

Here, the limit is $1+\frac{11}{3}$, or 5.

418. To determine the inferior limit to the negative roots, change the signs of the alternate terms; this will change the signs of the roots (Art. 400); then,

The *superior* limit of the roots of this equation, by changing its sign, will be the *inferior* limit of the roots of the proposed equation.

419. Proposition III.—*If the real roots of an equation, taken in the order of their magnitudes, be* a, b, c, d, *etc.*, a *being greater than* b, b *greater than* c, *and so on; then, if a series of numbers*, a′, b′, c′, d′, *etc.*, *in which* a′ *is greater than* a, b′ *a number between* a *and* b, c′ *a number between* b *and* c, *and so on, be substituted for* **x** *in the proposed equation, the results will be alternately positive and negative.*

The first member of the proposed equation is equivalent to $(x-a)(x-b)(x-c)(x-d)\ldots\ldots=0$.

Substituting for x the proposed series of numbers a', b', c', etc., we obtain the following results:

$(a'-a)(a'-b)(a'-c)(a'-d)$, etc. . $=+$ product, since *all* the factors are $+$.

$(b'-a)(b'-b)(b'-c)(b'-d)$, etc. . . . $=-$ product, since only one factor is $-$.

$(c'-a)(c'-b)(c'-c)(c'-d)$, etc. . $=+$ product, since two factors are $-$, and the rest $+$.

$(d'-a)(d'-b)(d'-c)(d'-d)$, etc. . $=-$ product, since an odd number of factors is $-$, and so on.

Corollary 1.—If two numbers be successively substituted for x, in any equation, and give results with *contrary* signs, there must be *one*, *three*, *five*, or some *odd* number of roots between these numbers.

Corollary 2.—If two numbers, substituted for x, give results with the *same* sign, then between these numbers there must be *two*, *four*, or some *even* number of real roots, or *no* roots at all.

Corollary 3.—If a quantity q, and every quantity greater thau q, render the results continually positive, q is greater than the greatest root of the equation.

Corollary 4.—Hence, if the signs of the alternate terms be changed, and if p, and every quantity greater than p, renders the result positive, then $-p$ is less than the least root of the equation.

ILLUSTRATION.—If we form the equation whose roots are 5, 2, and -3, the result is $x^3-4x^2-11x+30=0$. Now, if we substitute any number whatever for x, greater than 5, the result is *positive.* If we put $x=5$, the result is zero, as it should be.

If we substitute for x, any number less than 5, and greater than 2, the result is *negative.* Putting $x=2$, the result is zero.

Substituting for x, any number less than 2, and greater than -3, the result is *positive.* Substituting -3, it is zero.

Substituting a number less than -3, the result is negative.

From Cors. 3 and 4, it is easy to find when we have *passed all the real roots*, either in the ascending or descending scale.

STURM'S THEOREM.

420. *To find the number of real and imaginary roots of an equation.*

In 1834, M. Sturm gained the mathematical prize of the French Academy of Sciences, by the discovery of a beautiful theorem, by means of which the *number* and *situation* of all the real roots of an equation can, with certainty, be determiued. This theorem we shall now proceed to explain.

Let $X=x^n+Ax^{n-1}+Bx^{n-2}. \; . \; . \; . \; . \qquad +Tx+V=0$, be any equation of the n^{th} degree, containing no equal roots; for if the given equation contains equal roots, these may be found (Art. 414), and its degree diminished by division.

Let the first derived function of X (Art. 411) be denoted by X_1.

Divide X by X_1 until the remainder is of a lower degree with respect to x than the divisor, and call this remainder $-X_2$; that is, let the remainder, *with its sign changed*, be denoted by X_2.

Divide X_1 by X_2 in the same manner, and so on, as in the margin, denoting the successive remainders, with their *signs changed*, by X_3, X_4, etc., until we arrive at a remainder which does not contain x, which must always happen, since the equation having no equal roots, there can be no factor containing x. common to the equation and its first derived function. Let this remainder, having its sign changed, be called X_{r+1}.

$$\begin{array}{l} X_1)X\quad(Q_1 \\ \quad X_1Q_1 \\ \hline X-X_1Q_1=-X_2 \end{array}$$

$$\begin{array}{l} X_2)X_1\quad(Q_2 \\ \quad X_2Q_2 \\ \hline X_1-X_2Q_2=-X_3 \end{array}$$

$$\begin{array}{l} X_3)X_2\quad(Q_3 \\ \quad X_3Q_3 \\ \hline X_2-X_3Q_3=-X_4 \end{array}$$

In these divisions, we may, to avoid fractions, either multiply or divide the dividends and divisors by any *positive* number, as this will not affect the *signs* of the functions X, X_1, X_2, etc.

By this operation, we obtain the series of quantities

$$X,\ X_1,\ X_2,\ X_3.\ \ .\ \ .\ \ X_{r+1} \qquad (1).$$

Each member of this series is of a lower degree with respect to x than the preceding, and the last does not contain x. Call X the *primitive* function, and X_1, X_2, etc., *auxiliary* functions.

421. Lemma I.—*Two consecutive functions,* X_1, X_2, *for example, can not both vanish for the same value of* x.

From the process by which X_1, X_2, etc., are obtained, we have the following equations:

$$\begin{array}{ll} X\ \ =X_1Q_1-X_2. & \ldots\ldots\ (1) \\ X_1\ =X_2Q_2-X_3. & \ldots\ldots\ (2) \\ X_2\ =X_3Q_3-X_4. & \ldots\ldots\ (3) \\ & \\ X_{r-1}=X_rQ_r-X_{r+1}. & \ldots\ldots\ (r). \end{array}$$

If possible, let $X_1=0$, and $X_2=0$; then, by eq. (2) we have $X_3=0$; hence, by eq. (3) we have $X_4=0$; and proceeding in the same way, we shall find $X_5=0$, $X_6=0$, and finally $X_{r+1}=0$. But this is impossible, since X_{r+1} does not contain x, and therefore can not vanish for any value of x.

422. Lemma II.—*If one of the auxiliary functions vanishes for any particular value of* x, *the two adjacent functions must have contrary signs for the same value of* x.

Let us suppose that $X_3=0$, when $x=a$; then, because $X_2=X_3Q_3-X_4$, and $X_3=0$; therefore, $X_2=-X_4$; that is, X_2 and X_4 have contrary signs.

423. Lemma III.—*If any of the auxiliary functions vanishes when* x=a, *and* h *be taken so small that no root of any of the other functions in series* (1) *lies between* a—h *and* a+h, *then will the number of variations and permanences, when* a—h *and* a+h *are substituted for* x *in this series, be precisely the same.*

Suppose, for example, the substitution of a for x causes the function X_3 to vanish; then, by Art. 421, neither of the functions X_2 or X_4 can vanish for the same value of x; and since when X_3 vanishes, X_2 and X_4 have contrary signs, (Art. 422); therefore, the substitution of a for x in X_2, X_3, X_4, must give

$$\begin{array}{ccccccc} X_2, & X_3, & X_4, & \text{or} & X_2, & X_3, & X_4. \\ + & 0 & -, & & - & 0 & + \end{array}$$

And since h is taken so small that no root either of $X_2=0$, or $X_4=0$, lies between $a-h$ and $a+h$, the signs of these functions will continue the same whether we substitute $a-h$ or $a+h$ for x (Art. 419). Hence, whether we suppose X_3 to be + or — by the substitution of $a-h$ and $a+h$ for x, there will be *one* variation and *one* permanence. Thus, we shall have either

$$\begin{array}{ccccccc} X_2, & X_3, & X_4, & \text{or} & X_2, & X_3, & X_4. \\ + & \pm & - & & - & \pm & + \end{array}$$

So that no alteration in the number of variations and permanences can be made in passing from $a-h$ to $a+h$.

424. Lemma IV.—*If* a *is a root of the equation* X=0, *then the series of functions* X, X_1, X_2, *etc., will lose one variation of signs in passing from* a—h *to* a+h; h *being taken so small that no root of the function* $X_1=0$ *lies between* a—h *and* a+h.

For x substitute $a+h$ in the equation $X=0$, and denote the result by H. Also, put A, A′, A″ for the values of X and its derived functions when $a+h$ is substituted for x; then (Art. 411),

$$H=A+A'h+\tfrac{1}{2}A''h^2+, \text{ etc.}$$

But, since a is a root of the eq. $X=0$, we shall have $A=0$, while A′ can not be 0, since the eq. $X=0$ has no equal roots. Hence,

$$H=A'h+\tfrac{1}{2}A''h^2+, \text{ etc.}, =h(A'+\tfrac{1}{2}A''h+, \text{ etc}).$$

Now, h may be taken so small that the quantity within the parenthesis shall have the same sign as its first term A′, (since A′ expresses the first derived function of X, corresponding to X′, in Art. 412); therefore, the sign of X, when $x=a+h$, will be the *same* as the sign of X_1.

If we substitute $a-h$ for x in the equation $X=0$, and denote the result by H′, we then have, by changing h into $-h$, in the expression for H,

$$H'=-h(A'-\tfrac{1}{2}A''h+, \text{ etc}).$$

Now, it is evident that for very small values of h, the sign of H′ will depend upon the first term $-A'h$, and, consequently, will be *contrary* to that of A′. Hence, when $x=a-h$, there is a variation of signs in the first two terms of the series X, X_1; and when $x=a+h$, there is a continuation of the same sign. Therefore, *one variation is lost in passing from $x=a-h$ to $a+h$.*

If any of the auxiliary functions should vanish at the same time by making $x=a$, the number of variations will not be affected on this account (Art. 423), and therefore, one variation of signs will still be lost in passing from $a-h$ to $a+h$.

425. Sturm's Theorem.—*If any two numbers,* p *and* q, *(*p *being less than* q*) be substituted for* x *in the series of functions* X, X_1, X_2, *etc., the substitution of* p *for* x *giving* k *variations, and that of* q *for* x, *giving* k′ *variations; then,* k—k′ *will be the exact number of real roots of the equation* X=0, *which lie between* p *and* q.

Let us suppose that $-\infty$ is substituted for x, and suppose that x continually increases and passes through all degrees of magnitude till it becomes 0, and finally reaches $+\infty$.

Now, it is evident, that so long as x, with its minus sign, is less than any of the roots of $X=0$, $X_1=0$, etc., no alteration will take place in the signs of any of these functions (Art. 419); but when x becomes equal to the least root (with its sign) of any of the auxiliary functions, although a change may occur in the sign of this function, yet we have seen (Art. 423) that it is the *order* only, and not the *number* of variations which is affected. But when x becomes equal to any of the roots of the primitive function, then one variation of signs is always lost.

Since, then, a variation is always lost whenever the value of x passes through a root of the primitive function $X=0$, and since a variation can not be lost in any other way, nor can one be ever introduced, it follows that the excess of the number of variations given by $x=p$, above that given by $x=q$ $(p<q)$, is exactly equal to the number of real roots of $X=0$, which lie between p and q.

Corollary.—If the equation is of the n^{th} degree, and m represents the number of real roots, then (Art. 396), the number of imaginary roots will be $n-m$.

426. *To determine the number of real roots.*

Substitute $-\infty$ and $+\infty$ for x in the several functions, since the roots must all be comprised between these limits. In this case, the sign of each function will be that of its first term.

If we substitute 0 for x, the number of variations lost from $-\infty$ to 0, will be the number of *negative* roots; and from 0 to $+\infty$, the number of *positive* roots.

427. *To determine the situation of each real root; that is, the figures between which it lies.*

Substitute 0, —1, —2, —3, etc., for x, in series (1), till we find a number which produces as many variations as $x=-\infty$ produced. This will be the limit of the negative roots.

Substitute 1, 2, 3, etc., till we find a positive number which gives the same number of variations that $x=+\infty$ does. This will be the superior limit of the positive roots.

By observing where variations are lost, we find the situation of the roots. If two or more variations are lost between two of the substitutions, take smaller numbers, until only *one* is lost. This is termed the *separation of the roots.*

1. Find the number and situation of the real roots of the equation $4x^3-12x^2+11x-3=0$.

Here, we have $\quad X = 4x^3-12x^2+11x-3$,
and (Art. 411) $\quad X_1=12x^2-24x+11$.

Multiplying X by 3, to render the first term divisible by the first term of X_1, and proceeding as in the method of finding the G.C.D., (Art. 108), we have for a remainder $-2x+2$. Canceling the factor $+2$, and changing the signs (Art. 420), we have $X_2=x-1$. Dividing X_1 by X_2, we have for a remainder -1; therefore, $X_3=+1$. Hence,

$$X = 4x^3-12x^2+11x-3.$$
$$X_1=12x^2-24x+11.$$
$$X_2= x-1.$$
$$X_3=+1.$$

Put $-\infty$ and $+\infty$ for x, and we have (Art. 426), for

$x=-\infty$, $-\ +\ -\ +$ three variations, $\therefore k=3$.

$x=+\infty$, $+\ +\ +\ +$ no variation, $\therefore k'=0$.

$\therefore k-k'=3-0=3$, the number of real roots.

For $x=0$, $-\ +\ -\ +$, three variations $\therefore k=3$.

Hence, there is (Art. 426) no real root between 0 and $-\infty$. This we might also have learned from Art. 402, since there is no permanence in the proposed equation.

It is best to substitute integral numbers first, and afterward fractional, if two or more roots are found to lie between two whole numbers. Or, substitute fractions at once, thus:

	X	X_1	X_2	X_3	
For $x=-\infty$ the signs are	$-$	$+$	$-$	$+$	giving 3 var.
$x=0$	$-$	$+$	$-$	$+$	" 3 "
$x=+\frac{1}{4}$	$-$	$+$	$-$	$+$	" 3 "
$x=+\frac{1}{2}$	0	$+$	$-$	$+$	
$x=+\frac{3}{4}$ . .	$+$	$-$	$-$	$+$	" 2 "
$x=+1$.	0	$-$	0	$+$	

For $x=+1\frac{1}{4}$	$-$	$-$	$+$	$+$	giving 1 var.
$x=+1\frac{1}{2}$	0	$+$	$+$	$+$	
$x=+1\frac{3}{4}$	$+$	$+$	$+$	$+$	" 0 "
$x=+\infty$	$+$	$+$	$+$	$+$	" 0 "

The roots are $\frac{1}{2}$, 1, and $1\frac{1}{2}$. If these numbers had not been substituted, the loss of *one* variation in passing from $\frac{1}{4}$ to $\frac{3}{4}$; *one* in passing from $\frac{3}{4}$ to $1\frac{1}{4}$; and *one* in passing from $1\frac{1}{4}$ to $1\frac{3}{4}$, would have given the *situation* of the roots.

A careful study of this example will serve to illustrate the theorem. Thus, we see that there are *three* changes of sign of the primitive function, *two* of the first auxiliary function, and *one* of the second.

Again, while no variation is lost by the change of sign of either of the auxiliary functions, each change of sign of the primitive function occasions a loss of one variation.

2. How many real roots has the eq. $x^3-3x^2+x-3=0$?

Here, ... $X = x^3-3x^2+x-3$

$X_1=3x^2-6x+1$

$X_2= x+2$

$X_3=-25.$

For $x=-\infty$ the signs are $-\ +\ -\ -$, 2 variations, $\therefore$ k$=2$.

$x=+\infty$ the signs are $+\ +\ +\ -$, 1 variation, $\therefore$ k$'=1$.

$\therefore$ k$-$k$'=2-1=1$, the number of real roots.

One variation is lost in passing from 2 to 4 and $X=0$ when $x=3$; therefore, the root is $+3$.

Find the number and situation of the real roots in each of the following equations:

3. $x^3-2x^2-x+2=0$. Ans. Three. -1, $+1$, $+2$.

4. $8x^3-36x^2+46x-15=0$. Ans. Three. One between 0 and 1, one between 1 and 2, one between 2 and 3.

5. $x^3-3x^2-4x+11=0$. Ans. Three. One between -2 and -1, one between 1 and 2, one between 3 and 4.

6. $x^3-2x-5=0$. Ans. One between 2 and 3.

7. $x^3-15x-22=0$. Ans. Three. One root is -2, one between $-2\frac{1}{4}$ and $-2\frac{1}{2}$, one between 4 and 5.

8. $x^4-4x^3-3x+23=0$. Ans. Two. One between 2 and 3, and one between 3 and 4.

9. $x^4-2x^3-7x^2+10x+10=0$. Ans. Four. The limits are $(-3, -2)$; $(0, -1)$; $(2, 3)$; $(2, 3)$.

10. $x^5-10x^3+6x+1=0$. Ans. Five. The limits are $(-4, -3)$; $(-1, 0)$; $(-1, 0)$; $(0, 1)$; $(3, 4)$.

XIII. RESOLUTION OF NUMERICAL EQUATIONS.

428. In the preceding articles we have demonstrated the most important propositions in the theory of equations, and in some cases have shown how to find their roots.

The general solution of an equation higher than the fourth degree, has never yet been effected. In the practical application of Algebra, however, *numerical* equations most frequently occur; and when the roots of these are real, they can always be found, either exactly or approximately. The way for doing this has been prepared in the preceding articles, by finding the limits of the roots, and separating them from each other.

RATIONAL ROOTS.

429. Proposition I.—*To determine the integral roots of an equation.*

If a be an integral root of the equation $Ax^4+Bx^3+Cx^2+Dx+E=0$, we shall have $Aa^4+Ba^3+Ca^2+Da+E=0$; therefore, $\frac{E}{a}=-Aa^3-Ba^2-Ca-D$.

Now, since the second member of the last equation is evidently a whole number, E is divisible by a. Put $\frac{E}{a}=E'$; transpose D to the first member, and divide by a; this gives

$$\frac{E'+D}{a}=-Aa^2-Ba-C;\ \therefore\ a \text{ is also a divisor of } E'+D.$$

Put $\frac{E'+D}{a}=D'$, transpose C, and divide by a; this gives

$$\frac{D'+C}{a}=-Aa-B;\ \therefore\ a \text{ is a divisor of } D'+C.$$

Again, put $\frac{D'+C}{a}=C'$, transpose B, divide by a, and $\frac{C'+B}{a}=-A$.

Lastly, making $\frac{C'+B}{a}=B'$, and transposing A, we have $B'+A=0$.

If, then, *all* these conditions are satisfied, a is a root of the proposed equation; but if any one of them fails, a is not a root. Hence, we have the following

Rule for finding the Integral Roots of an Equation.—*Divide the last term of the equation by any of its divisors* a, *and add to the quotient the coëfficient of the term containing* x. *Divide this sum by* a, *and add to the quotient the coëfficient of* x^2.

Proceed in this manner unto the first term, and if a *be a root, all these quotients will be whole numbers, and the result will be* 0.

Corollary 1.—It will be easier to ascertain whether $+1$ and -1 are roots, by trial. Also, by ascertaining the limits to the positive and negative roots (Art. 417), we may reduce the number of divisors.

Corollary 2.—If the first coëfficient be not unity, the equation may have a fractional root. To determine if this be the case, transform the equation into one having its first coëfficient unity (Art. 405, Cor. 1), and its roots integers (Art. 399).

Corollary 3.—When all the roots except two are integral, divide the equation and find the others (Art. 396, Cor. 1).

1. Find the rational roots of the equation

$$x^3+3x^2-4x-12=0.$$

Here, by Art. 417, no positive root can exceed $1+\sqrt[3]{12}$, or 4, and the limit of the negative roots is $1+3=4$.

It is also found, by trial, that $+1$ and -1 are not roots.

We then proceed to arrange the divisors of -12, among which it is *possible* to find the roots, and proceed as follows:

Last term			-12			
Divisors	$+2$,	$+3$,	$+4$,	-2,	-3,	-4
Quotients	-6,	-4,	-3,	$+6$,	$+4$,	$+3$
Add -4	-10,	-8,	-7,	$+2$,	-0,	-1
Quotients	-5,	*,	*,	-1,	0	*
Add $+3$	-2,			$+2$,	$+3$,	
Quotients	-1			-1,	-1,	
Add $+1$	0,			0,	0,	

Since -8, -7, and -1, are not divisible by $+3$, $+4$, and -4, we proceed no further with these divisors, as it is evident that they are not roots of the equation. The roots are $+2$, -2, and -3.

Find the roots of the following equations:

2. $x^3-7x^2+36=0$. . . . . . Ans. 3, 6, and -2.

When any term is wanting, as the 3d term in this example, its place must be supplied with 0. When there are equal roots, they may be found (Art. 414), or having found one, reduce the degree of the equation by division, and proceed as before.

3. $x^3-6x^2+11x-6=0$. . . . . . . Ans. 1, 2, 3.

4. $x^3+x^2-4x-4=0$. . . . . . Ans. 2, -1, -2.

5. $x^3-3x^2-46x-72=0$. . . . Ans. 9, -2, -4.

6. $x^3-5x^2-18x+72=0$. . . . Ans. 3, 6, -4.

7. $x^4-10x^3+35x^2-50x+24=0$. Ans. 1, 2, 3, 4.

8. $x^4+4x^3-x^2-16x-12=0$. Ans. $2, -1, -2, -3$.
9. $x^4-4x^3-19x^2+46x+120=0$. Ans. $4, 5, -2, -3$.
10. $x^4-27x^2+14x+120=0$. Ans. $3, 4, -2, -5$.
11. $x^4+x^3-29x^2-9x+180=0$. Ans. $3, 4, -3, -5$.
12. $x^3-2x^2-4x+8=0$. Ans. $2, 2, -2$.
13. $x^3+3x^2-8x+10=0$. Ans. $-5, 1\pm\sqrt{-1}$.
14. $x^4-9x^3+17x^2+27x-60=0$. Ans. $4, 5, \pm\sqrt{3}$.
15. $2x^3-3x^2+2x-3=0$. Ans. $\frac{3}{2}, \pm\sqrt{-1}$.
16. $3x^3-2x^2-6x+4=0$. Ans. $\frac{2}{3}, \pm\sqrt{2}$.
17. $8x^3-26x^2+11x+10=0$. Ans. $\frac{5}{2}, \frac{1}{8}(3\pm\sqrt{41})$.
18. $6x^4-25x^3+26x^2+4x-8=0$. Ans. $2, 2, \frac{2}{3}, -\frac{1}{2}$.
19. $x^4-9x^3+\frac{45}{4}x^2+\frac{27}{2}x-\frac{81}{4}=0$. Ans. $\frac{3}{2}, \frac{3}{2}, 3\pm3\sqrt{2}$.

IRRATIONAL ROOTS—METHODS OF APPROXIMATION.

Having found all the integral roots, we must have recourse to methods of approximation, the best of which is Horner's.

430. Horner's Method of Approximation.—The principle of this method depends on the successive transformations of the given equation, by Synthetic Division (Art. 410), so as to diminish its roots at each step of the operation.

Let the equation, one of whose roots is to be found, be

$$Px^n+Qx^{n-1}\ldots\ldots+Tx+V=0.$$

Suppose a to be the integral part of the root required, and r, s, t . . the decimal digits taken in order, so that $x=a+r+s+t\ldots$ Find a by trial, or by Sturm's theorem, and transform the equation into one whose roots shall be diminished by a (Art. 410).

Let $Py^n+Q'y^{n-1}\ldots\ldots+T'y+V'=0$ be the transformed equation; then, the value of y is the decimal $r+s+t\ldots\ldots$; and since

this root is contained between 0 and 1, we may easily find its first digit r. Again, let the roots of this equation be diminished by r, and let the transformed equation be

$$Pz^n+Q''z^{n-1}. \ . \quad . \ . \ . \ . \ +T''z+V''=0.$$

Now, the value of z in this equation is $s+t.$ . . . , and the value of s lies between .00 and .1; that is, it is either .00, .01, .02, . or .09. But since the figure s is in the second place of decimals, z^2, z^3 . . will be small, and we may generally find s from the equation $T''z+V''=0$; or, $s=-V\div T$, nearly.

Having found s, diminish the roots of the last equation by s, and then from the last two terms, $T'''z'+V'''$, of the resulting equation, find t the next decimal figure, and so on.

431. The absolute number, or last term, is sometimes called the *dividend*, and the coëfficient of the first power of the unknown quantity, (as, T'' or T''',) the *incomplete* or *trial divisor*.

The correctness of the values of s, t, etc., obtained by means of the trial divisor, will always be verified in the next operation. If too great or too small, the quotient figure must be *increased* or *diminished*.

The accuracy with which each succeeding decimal figure may be found, increases as the value of the figure decreases. In general, after finding three or four decimal figures, the rest may be obtained with sufficient accuracy by dividing V'' by T''.

432. By changing the signs of the alternate terms (Art. 400), and finding the *positive* roots of the resulting equation, we may obtain the *negative* roots of the proposed equation.

REMARK.—It is generally easier to find the first decimal figure of the root by trial than by Sturm's theorem.

433. To illustrate this method, let it be required to find the positive root of the equation $x^2-4x-10.768649=0$.

We readily find that x must be greater than 5, and less than 6;

therefore, $a=5$. We then proceed to transform this equation into another whose roots shall be less by 5. (See Art. 410.)

$$\begin{array}{llrr}
 & a & & \\
 & 5) & 1-4 & -10.768649 \\
 & & +5 & +\ 5 \\
\cline{3-4}
 & & +1 & -\ 5.768649 \\
 & & +5 & \\
\cline{3-3}
 & & +6 & \\
\cline{3-3}
\text{1st Trans. eq.} \ldots & & y^2+6y & -\ 5.768649=0.
\end{array}$$

Here we may find the value of y nearly, by dividing 5.7 by 6, which gives .9; but this is too great, because we neglected y^2. If we assume $y=.8$, and deduct $y^2=.64$ from 5.7, and then divide by 6, we see that y must be .8. Let us now transform the equation into another whose roots shall be less by .8.

$$\begin{array}{lllrr}
 & s & & & \\
 & .8) & 1 & +6 & -5.768649 \\
 & & & .8 & +5.44 \\
\cline{4-5}
 & & & +6.8 & -\ .328649 \\
 & & & .8 & \\
\cline{4-4}
 & & & 7.6 & \\
\cline{4-4}
\text{2d Trans. eq.} \ldots\ldots & & & \multicolumn{2}{l}{z^2.+7.6z-.328649=0.}
\end{array}$$

The approximate value of z in this equation is the second decimal figure of the root. This is readily found by dividing the absolute term by the coëfficient of z, the first term, z^2, being now so small that it may be neglected. Thus, $.328\div7.6=.04=s$.

We next diminish the roots of the last equation by .04.

$$\begin{array}{lllrr}
 & s & & & \\
 & .04) & 1 & +7.6 & -.328649 \\
 & & & .04 & .3056 \\
\cline{4-5}
 & & & +7.64 & .023049 \\
 & & & .04 & \\
\cline{4-4}
 & & & +7.68 & \\
\text{3d Trans. eq.} \ldots\ldots & & & z'^2+7.68z' & -.023049=0.
\end{array}$$

Here z' is nearly $.023 \div 7.68 = .003 = t$.

By diminishing the roots of the last equation by .003, we have

$$\begin{array}{llll}
t & & & \\
.003) & 1 & +7.68 & -.023049 \\
 & & .003 & .023049 \\
\hline
 & & +7.683 & .0
\end{array}$$

The remainder being zero, shows that we have obtained the exact root, which is 5.843.

By changing the sign of the second term of the proposed equation, we have $x^2+4x-10.768649=0$. The root of this equation may be found in a similar manner; it is 1.843. Hence, the two roots are $+5.843$ and -1.843.

Ex. 2.—To illustrate this method further, let us form the equation whose roots are 3, $+\sqrt{2}$, $-\sqrt{2}$, which gives $x^3-3x^2-2x+6=0$. Let it now be required to find, by Horner's method, the root which lies between 1 and 2; that is, $\sqrt{2}$.

One root lies between 1 and 2; hence, $a=1$, and the first step is to transform the equation so as to diminish its roots by 1.

$$\begin{array}{llrrr}
a & & & & \\
1) & 1 & -3 & -2 & +6 \\
 & & +1 & -2 & -4 \\
\hline
 & & -2 & -4 & +2 \\
 & & +1 & -1 & \\
\hline
 & & -1 & -5 & r=\frac{V'}{T'}=\frac{2}{5}=.4 \\
 & & +1 & & \\
\hline
 & & 0 & &
\end{array}$$

Hence, $y^3 \pm 0y^2-5y+2=0$, is the first transformed equation. By dividing the absolute term 2 by 5, the trial divisor or coëfficient of y, we find $r=.4$, and proceed to transform the equation so as to diminish its roots by .4.

$$\begin{array}{lllll}
r & & & & \\
.4) & 1 & \pm 0 & -5 & +2 \\
 & & \underline{\quad .4} & \underline{\quad .16} & \underline{-1.936} \\
 & & \quad .4 & -4.84 & +\ .064 \\
 & & \underline{\quad .4} & \underline{+\ .32} & \\
 & & \quad .8 & -4.52 & s=\frac{V''}{T''}=\frac{.064}{4.52}=.01 \\
 & & \underline{\quad .4} & & \\
 & & \quad 1.2 & &
\end{array}$$

This gives $z^3+1.2z^2-4.52z+.064=0$, for the 2d transformed equation; and for s, the next figure of the root, .01.

Transform this equation so as to diminish its roots by .01.

$$\begin{array}{lllll}
s & & & & \\
.01) & 1 & +1.2 & -4.52 & +.064 \\
 & & \underline{\quad .01} & \underline{\quad .0121} & \underline{-.045079} \\
 & & \quad 1.21 & -4.5079 & +.018921 \\
 & & \underline{\quad .01} & \underline{\quad .0122} & \\
 & & \quad 1.22 & -4.4957 & \\
 & & \underline{\quad .01} & & t=\frac{V'''}{T'''}=\frac{.0189}{4.495}=.004 \\
 & & \quad 1.23 & &
\end{array}$$

This gives $z'^3+1.23z'^2-4.4957z'+.018921=0$, for the 3d transformed equation; and for the next figure of the root $t=.004$.

Transform this equation so as to diminish its roots by .004.

$$\begin{array}{lllll}
t & & & & \\
.004) & 1 & +1.23 & -4.4957 & +.018921 \\
 & & \underline{\quad .004} & \underline{+\ .004936} & \underline{-.017963056} \\
 & & \quad 1.234 & -4.490764 & \quad .000957944 \\
 & & \underline{\quad .004} & \underline{+\ .004952} & \\
 & & \quad 1.238 & -4.485812 & \\
 & & \underline{\quad .004} & & \\
 & & \quad 1.242 & &
\end{array}$$

We may obtain several of the succeeding figures accurately by division; thus, $.000957944 \div 4.485812 = .0002135$, which is true to the last decimal place, as will be found by extracting the square root of 2. Hence, $x=1.4142135$.

In practice it is customary to make some abridgments. Thus, mark with a * the coëfficients of the unknown quantity in each transformed equation instead of rewriting it. Also, when the root is required only to five or six places of decimals, use about this number in the operation.

3. Given $x^4-8x^3+14x^2+4x-8=0$, to find a value of x.

OPERATION.

5.236068)	—8	+14	+4	—8
	5	—15	—5	—5
	—3	— 1	—1	*—13
	5	10	45	10.6576
	2	9	*44	*— 2.3424
	5	35	9.288	1.93880241
	7	*44	53.288	*— .40359759
	5	2.44	9.784	.39905490
	*12	46.44	*63.072	*— .00454269
	.2	2.48	1.554747	.00400954
	12.2	48.92	64.626747	*— .00053315
	.2	2.52	1.566321	
	12.4	*51.44	*66.193068	
	.2	.3849	.31608	
	12.6	51.8249	66.50915	
	.2	.3858	.31656	
	*12.8	52.2107	*66.82571	
	.03	.3867		
	12.83	*52.5974		
	.03	.08		
	12.86	52.68		
	.03	.08		
	12.89	52.76		
	.03			
	*12.92			

As the root is found only to six decimal places, carry the *true divisor* for the third figure (6) to five decimal places. This divisor is 66.50915, which, multiplied by .006, gives eight decimal places; and the dividend ought to be carried thus far, to make the figure in the sixth decimal place of the root correct.

The divisor, 66.825, for the fifth figure of the root, requires to be carried only to three decimal places, for the product of this number by .00006 gives eight decimal places, as it ought to do. So the divisor for the last figure (8) of the root would require to be carried only to two decimal places.

The numbers in the preceding columns require to be carried to still fewer places, as will readily be perceived.

The last three figures of the root may be obtained merely by division; thus, $.00454269 \div 66.82571 = .000068$, nearly.

Observe that where decimals are omitted, we always take the figure next to the omitted places, *to the nearest unit.* Thus, .07752 is nearer .08 than .07; therefore, the former is taken.

434. Horner's method may be applied to equations of any degree, and is the most elegant method of approximation yet discovered. It may be expressed by the following

Rule.—1. *Find, by trial, or by Sturm's theorem, the integral part of the required root.*

2. *Transform the equation* (Art. 410) *into another whose roots shall be those of the proposed equation, diminished by the part of the root already found.*

3. *With the absolute term in the first transformed equation for a dividend, and the coëfficient of* x *for a divisor, find the first decimal figure of the root.*

4. *Transform the last equation into another whose roots shall be diminished by the part of the root already found, and from the first two terms of this equation, find the second figure of the root.*

5. *Continue this process till the root is found to the required degree of accuracy.*

6. *To find the negative roots, change the signs of the alternate terms, and proceed as for a positive root.*

REMARKS.—1. If any figure, found by trial, is too great or too small, it will be made manifest in the next transformation. (See Art. 431.)

2. After finding three figures of the root, the next three may generally be obtained by dividing the absolute term by the coëfficient of x.

Find at least one value of x in each of the following:

1. $x^2+5x-12.24=0$. . . . . Ans. $x=1.8$.
2. $x^2+12x-35.4025=0$. . . . Ans. $x=2.45$.
3. $4x^2-28x-61.25=0$. . . . . Ans. $x=8.75$.
4. $8x^2-120x+394.875=0$. . . Ans. $x=10.125$.
5. $5x^2-7.4x-16.08=0$. . . . . . Ans $x=2.68$.
6. $x^2-6x+6=0$. . . . . Ans. $x=4.73205$.
7. $x^3+4x^2-9x-57.623625=0$. . Ans. $x=3.45$.
8. $2x^3-50x+32.994306=0$. . Ans. $x=4.63$.
9. $x^3+4x^2-5x-20=0$. . . . Ans. $x=2.23608$.
10. $x^3-2x-5=0$. . . . Ans. $x=2.0945515$.
11. $x^3+10x^2-24x-240=0$. Ans. $x=4.8989795$.
12. $x^4-8x^3+20x^2-15x+.5=0$. Ans. $x=1.284724$.
13. $x^4-59x^2+840=0$. . . Ans. $x=4.8989795$.
14. $2x^4+5x^3+4x^2+3x=8002$. Ans. $x=7.335554$.
15. $x^5+2x^4+3x^3+4x^2+5x=54321$. A. $x=8.414455$.

435. *To extract the roots of numbers by Horner's Method.*

This is only a particular case of the solution of the equation $x^n=N$, or $x^n-N=0$; an equation of the n^{th} degree, in which all the terms are wanting except the first and last.

In performing the operation, observe that the successive integral figures have the same relation to each other that the successive decimal places have in the previous examples.

In extracting any root, point off the given number into periods, as in the operation by the common rule.

For an example, let it be required to find the cube root of 12977875; that is, one root of the equation $x^3-12977875=0$.

235)	1	0	0	12977875
		2	4	8
		2	4	4977
		2	8	4167
		4	*12	810875
		2	189	810875
	1	*6	1389	
		3	198	
		63	*1587	
		3	3475	
		66	162175	
		3		
	1	*69		
		5		
		695		

The reason for placing the figures as they are in the successive columns, will be readily understood by using the numbers 200 and 30, instead of 2 and 3.

By the same method find

2. The cube root of 34012224. . . . Ans. 324.
3. The cube root of 9. . . . . Ans. 2.080084.
4. The cube root of 30. . . . Ans. 3.107233.
5. The fifth root of 68641485507. . . Ans. 147.

APPROXIMATION BY DOUBLE POSITION.

436. Double Position furnishes one of the most useful methods of approximating to the roots of equations. It has the advantage of being applicable, whether the equation is *fractional*, *radical*, or *exponential*, or to any other form of function.

Let $X=0$, represent any equation; and suppose that a and b, substituted for x, give results, the one too *small*, and the other too *great*, so that one root lies between a and b. (Art. 403.)

Let A and B be the results arising from the substitution of a and b for x, in the equation $X=0$. Let $x=a+h$, and $b=a+k$; then, if we substitute $a+h$ and $a+k$ for x, in the equation $X=0$, we shall have

$$X=A+A'h+\tfrac{1}{2}A''h^2+, \text{ etc.}$$
$$B=A+A'k+\tfrac{1}{2}A''k^2+, \text{ etc.}$$

Here, A', A'', etc., are the derived functions of A (Art. 411). Now, if h and k be so small that their second and higher powers may be neglected without much error, we shall have

$$X-A=A'h \text{ nearly;}$$
$$B-A=A'k \quad \text{"}$$

Whence, $B-A : X-A :: A'k : A'h :: k : h$;
Or, . . $B-A : k :: X-A : h$, (Art. 270);
Or, $B-A : b-a :: X-A : h$, since $k=b-a$.

Hence, we have the following

Rule.—*Find, by trial, two numbers which, substituted for x, give one a result too small, and the other too great. Then say, As the difference of the results is to the difference of the suppositions, so is the difference between the true and the first result, to the correction to be added to the first supposition.*

Substitute this approximate value for the unknown quantity, and find whether it is too small or too great; then, take two less numbers, such that the true root may lie between them, and proceed as before, and so on.

It is generally best to begin with two integers which differ from each other by unity, and to carry the first approximation only to *one* place of decimals. In the next operation make the difference of the suppositions 0.1, and carry the 2d quotient to *two* places, and so on.

1. Given $x^3+x^2+x=100$, to find x.

Here, x lies between 4 and 5. Substitute these two numbers for x in the given equation, and the result is as follows:

4	x	5	
64	x^3	125	
16	x^2	25	
4	x	5	
84	results	155	
155	5	100	
84	4	84	
71 :	1 ::	16	: 0.22,

therefore, $x=4.2$, the first approximation.

Substituting 4.2 and 4.3 for x, and proceeding as before, we get for a second approximation $x=4.264$. Assuming $x=4.264$ and 4.265, and continuing, we obtain $x=4.2644299$, nearly.

Find one root of each of the following equations:

2. $x^3+30x=420$. . . . . . . Ans. $x=6.170103$.
3. $144x^3-973x=319$. . . . . . . Ans. $x=2.75$.
4. $x^3+10x^2+5x=2600$. . . Ans. $x=11.00679$.
5. $2x^3+3x^2-4x=10$. . . . . Ans. $x=1.62482$.
6. $x^4-x^3+2x^2+x=4$. . . . . Ans. $x=1.14699$.
7. $\sqrt[3]{7x^3+4x^2}+\sqrt{10x(2x-1)}=28$. A. $x=4.51066$.

437. Newton's Method of Approximation.—This method, now but little used, is briefly as follows:

Find, by trial, a quantity a within less than 0.1 of the value of the root. Substitute $a+y$ for x in the given equation, and it will be of this form

$$A+A'y+\tfrac{1}{2}A''y^2+\tfrac{1}{6}A'''y^3+, \text{ etc.}, =0 \quad \text{(Art. 411)},$$

where A, A′, A″, etc., are what the proposed equation, the first derived polynomial, etc., become when $x=a$.

From this equation, by transposing and dividing,

We find $y=-\frac{A}{A'}-\frac{1}{2}\frac{A''}{A'}y^2-\frac{1}{6}\frac{A'''}{A'}y^3-$, etc.; and since

y is <0.1, y^2 will be <0.01, $y^3<0.001$, and so on.

Therefore, if the sum of the terms containing y^2, y^3, etc., be less than .01, we shall, in neglecting them, obtain a value of y within .01 of the truth. Putting $y=-\frac{A}{A'}$, we have $x=a-\frac{A}{A'}$. This will differ from the true value of x by less than .01.

Now, put b for this approximate value of x, and let $x=b+z$; we have then as before

$$B+B'z+\tfrac{1}{2}B''z^2+\tfrac{1}{6}B'''z^3+\text{, etc., }=0;$$

and as z is supposed to be less than .01, z^2 will be $<.0001$. If, then, we neglect the terms containing z^2, z^3, etc., we shall obtain a probable value of z within .0001; and so on. Applying the successive corrections, we obtain the value of x.

Newton gave but a single example, viz.

Required to find the value of x in the equation $x^3-2x-5=0$. Ans. $x=2.09455149$.

CARDAN'S RULE FOR SOLVING CUBIC EQUATIONS.

438. In its most general form, a cubic equation may be represented by

$$x^3+px^2+qx+r=0;$$

but as we can always take away the second term, (Art. 407,) we will suppose, to avoid fractions, that it is reduced to the form

$$x^3+3qx+2r=0.$$

Assume $x=y+z$, and the equation becomes

$$y^3+z^3+3yz(y+z)+3q(y+z)+2r=0.$$

Now, since we have two unknown quantities in this equation, and have made only one supposition respecting them, we are at liberty

to make another. Let, therefore, $yz=-q$. Substituting, we have $y^3+z^3+2r=0$; but since $yz=-q$, $z^3=-\frac{q^3}{y^3}$;

Hence, ... $y^3-\frac{q^3}{y^3}+2r=0$;

Whence, ... $y^3=-r+\sqrt{r^2+q^3}$.

And similarly, . . $z^3=-r-\sqrt{r^2+q^3}$;

the radical being positive in one, and negative in the other, by reason of the relation $yz=-q$.

And since $x=y+z$, we have

$$x=\sqrt[3]{(-r+\sqrt{r^2+q^3})}+\sqrt[3]{(-r-\sqrt{r^2+q^3})}.$$

This formula will give one of the roots. The others may be found by reducing the equation (Art. 396, Cor. 1).

439. If r^2+q^3 be negative, that is, if $r^2+q^3<0$, the values of x become apparently imaginary when they are actually real, and we shall now show that

Cardan's Method of Solution does not extend to those cases in which the equation has three real and unequal roots.

Suppose the one real root (Art. 401, Cor. 3), to be a; and the other two arising from the solution of a quadratic to be $b+\sqrt{3c}$, and $b-\sqrt{3c}$, in which, if $3c$ be positive, the roots are real, and if $3c$ be negative, they are imaginary; and because the second term of the equation is 0, we have (Art. 398),

$$0=a+(b+\sqrt{3c})+(b-\sqrt{3c})=a+2b;$$
$$3q=a\times 2b+b^2-3c=-3b^2-3c;$$
$$2r=-a(b^2-3c)=2b^3-6bc.$$

Hence, we have

$$r^2+q^3=(b^3-3bc)^2-(b^2+c)^3=-9b^4c+6b^2c^2-c^3$$
$$=-c(3b^2-c)^2 \therefore \sqrt{r^2+q^3}=(3b^2-c)\sqrt{-c}.$$

Now, this expression is real when c is negative, and imaginary when c is positive, or when the equation has three real roots.

If $c=0$, the roots are a, b, and b; hence, Cardan's Rule is applicable to equations containing two equal roots.

440. In illustration of the apparent paradox that when the roots of the quadratic equation, Art. 438, are imaginary, the roots of the cubic equation are all real, take the following

EXAMPLE.—To find the three roots of the equation $x^3-15x-4=0$.

By substituting $y+z$ for x, we have

$$y^3+z^3+3yz(y+z)-15(y+z)-4=0;$$

And, since . . $3yz=15$, $y^3+z^3-4=0$.

From the solution of these equations, we obtain $y^3=2+11\sqrt{-1}$. By actual multiplication, we find that $y=2+\sqrt{-1}$; likewise, $z^3=2-11\sqrt{-1}$, and $z=2-\sqrt{-1}$.

Hence, $x=y+z=(2+\sqrt{-1})+(2-\sqrt{-1})=4$.

By dividing the given equation by $x-4$, we find the other two roots are $x=-2+\sqrt{3}$, and $-2-\sqrt{3}$.

As no means have yet been discovered for reducing the imaginary forms to real values, Cardan's rule fails when all the roots are real. This is the *Irreducible Case* of cubic equations.

441. The following examples, containing one real and two imaginary roots, may be solved by Cardan's rule.

When the equation contains the second term, remove it (Art. 407), and reduce the equation to the form $x^3+3qx+2r=0$.

Then, $x=\sqrt[3]{(-r+\sqrt{r^2+q^3})}+\sqrt[3]{(-r-\sqrt{r^2+q^3})}$, will be the real root of the proposed equation.

Having the real root, the imaginary roots may be found by reducing the equation to a quadratic (Art. 396, Cor. 1).

1. Solve the equation $v^3+3v^2+9v-13=0$.

Substituting $x-1$ for v (Art. 407), we have $x^3+6x-20=0$.

Comparing this with the equation $x^3+3qx+2r=0$, we find $q=2$, $r=-10$; hence,

$$x=\sqrt[3]{(10+\sqrt{108})}+\sqrt[3]{(10-\sqrt{108})}=2.732-.732=2.$$

Whence, $v=x-1=2-1=1$.

The other two roots are easily found to be $-1\pm3\sqrt{-1}$.

2. $x^3-9x+28=0$. . . . Ans. $x=-4,\ 2\pm\sqrt{-3}$.
3. $x^3+6x-2=0$. . . Ans. $x=\sqrt[3]{4}-\sqrt[3]{2}=.32748$.
4. $x^3-6x^2+13x-10=0$. Ans. $x=2,\ 2\pm\sqrt{-1}$.
5. $x^3-9x^2+6x-2=0$. . . . . Ans. $x=8.306674$.

REMARK.—Cardan's Rule, together with those of Ferrari, Euler, Descartes, and others, are regarded, since the discovery of Horner's method, and Sturm's theorem, as little more than analytical curiosities.

RECIPROCAL OR RECURRING EQUATIONS.

442. A **Recurring or Reciprocal Equation** is one such that if a be one of its roots, the reciprocal of a will be another.

Proposition I.—*In a recurring equation the coëfficients, when taken in a direct and in an inverse order, are the same.*

Let $x^n+Ax^{n-1}+Bx^{n-2}. \ldots +Sx^2+Tx+V=0$, be a recurring equation; that is, one that is satisfied by the substitution of the reciprocal of x for x. This gives

$$\frac{1}{x^n}+\frac{A}{x^{n-1}}+\frac{B}{x^{n-2}}.\ \ldots\ldots+\frac{S}{x^2}+\frac{T}{x}+V=0;$$

and multiplying by x^n,

$$1+Ax+Cx^2.\ \ldots\ldots +Sx^{n-2}+Tx^{n-1}+Vx^n=0,$$

which proves the proposition.

Such equations are called *Recurring Equations*, from the forms of their *coëfficients;* and *Reciprocal Equations*, from the forms of their *roots.*

Proposition II.—*A recurring equation of an odd degree, has one of its roots equal to* $+1$, *when the signs of the like coëfficients are different, but equal to* -1, *when their signs are alike.*

Since every power of $+1$ is positive; when the signs of the like coëfficients are *different*, if we substitute $+1$ for x the corresponding terms will destroy each other.

When the signs of the like coëfficients are the *same*, since one will belong to an *odd*, and the other to an *even* power, if we substitute -1 for x, the corresponding terms will destroy each other.

Such equations may, therefore, be reduced one degree lower by dividing by $x-1$, or $x+1$.

Proposition III.—*A recurring equation of an even degree, whose like coëfficients have opposite signs, is divisible by* x^2-1, *and therefore two of its roots are* $+1$, *and* -1.

Let $x^{2n}+Ax^{2n-1}+Bx^{2n-2}\ldots\ldots-Bx^2-Ax-1=0$, be an equation of the kind specified. It may evidently be arranged thus, and be divisible by x^2-1 (Art. 83).

$$(x^{2n}-1)+Ax(x^{2n-2}-1)+Bx^2(x^{2n-4}-1)+, \text{ etc.} \ldots=0.$$

Corollary.—An equation of this form may therefore be reduced two degrees lower by either common or synthetic division.

Proposition IV.—*Every recurring equation of an even degree above the second, may be reduced to an equation of half that degree, when the signs of the corresponding terms are alike.*

For, $x^{2n}-Ax^{2n-1}+Bx^{2n-2}-$, etc. $\ldots\ldots+Bx^2-Ax+1=0$, by

dividing by x^n, and collecting the pairs of terms equi-distant from the extremes, becomes of the form

$$\left(x^n+\frac{1}{x^n}\right)-A\left(x^{n-1}+\frac{1}{x^{n-1}}\right)+B.\left(x^{n-2}+\frac{1}{x^{n-2}}\right)-, \text{ etc.}, =0.$$

Let $x+\frac{1}{x}=z$; then, $x^2+\frac{1}{x^2}=z^2-2$, by squaring; also,

$$\left(x^3+\frac{1}{x^3}\right)=\left(x^2+\frac{1}{x^2}\right)z-\left(x+\frac{1}{x}\right)=(z^2-2)z-z;$$

and generally $\left(x^n+\frac{1}{x^n}\right)=\left(x^{n-1}+\frac{1}{x^{n-1}}\right)z-\left(x^{n-2}+\frac{1}{x^{n-2}}\right).$

Hence, each of the binomials may be expressed in terms of z, and the resulting equation will be of the n^{th} degree.

1. Given $x^4-5x^3+6x^2-5x+1=0$, to find x.

Here, ... $x^2-5x+6-\frac{5}{x}+\frac{1}{x^2}=0,$

Or, ... $\left(x^2+\frac{1}{x^2}\right)-5\left(x+\frac{1}{x}\right)+6=0.$

Let $x+\frac{1}{x}=z$; then, $z^2-5z+4=0$, and $z=4$, or 1;

Putting $x+\frac{1}{x}$ equal to each of these values of z, we obtain for the four values of x, $2\pm\sqrt{3}$ and $\frac{1}{2}(1\pm\sqrt{-3})$.

The second of these values is the reciprocal of the first, and the fourth of the third, as may be shown, thus:

$$\frac{1}{2+\sqrt{3}}=\frac{1}{2+\sqrt{3}}\times\frac{2-\sqrt{3}}{2-\sqrt{3}}=\frac{2-\sqrt{3}}{4-3}=2-\sqrt{3}.$$

EXAMPLES IN RECURRING EQUATIONS.

1. $x^4-10x^3+26x^2-10x+1=0.$

Ans. $x=3\pm2\sqrt{2},\ 2\pm\sqrt{3}.$

2. $x^4-\frac{5}{2}x^3+2x^2-\frac{5}{2}x+1=0.$

Ans. $x=2,\ \frac{1}{2},\ \pm\sqrt{-1}.$

3. $x^4-3x^3+3x-1=0$. Ans. $x=\pm1,\ \frac{1}{2}(3\pm\sqrt{5})$.

4. $x^5-11x^4+17x^3+17x^2-11x+1=0$.

Ans. $x=-1,\ \frac{9+\sqrt{77}}{2},\ \frac{9-\sqrt{77}}{2},\ \frac{3+\sqrt{5}}{2},\ \frac{3-\sqrt{5}}{2}$.

5. $4x^6-24x^5+57x^4-73x^3+57x^2-24x+4=0$.

Ans. $x=2,\ \frac{1}{2},\ 2,\ \frac{1}{2},\ \frac{1+\sqrt{-3}}{2},\ \frac{1-\sqrt{-3}}{2}$.

BINOMIAL EQUATIONS.

443. Binomial equations are those of the form

$$y^n\pm A=0.$$

Let . . . . $\sqrt[n]{A}=a$; that is, $A=a^n$;

Then, ... $y^n\pm a^n=0$.

Let . . . $y=ax$; then, $a^nx^n\pm a^n=0$,

Or, . . . $x^n\pm1=0$,

which is a recurring equation.

444.—I. The roots of the equation $x^n\pm1=0$, are all unequal; for the first derived polynomial nx^{n-1}, evidently has no divisor in common with $x^n\pm1$, and therefore there are no equal roots (Art. 414).

II.—If n be even, the equation $x^n-1=0$, or $x^n=1$, has two real roots, $+1$ and -1, and no more, because no other real number can, by its involution, produce 1.

By dividing $x^n-1=0$ by $(x+1)(x-1)=x^2-1$, we have

$$x^{n-2}+x^{n-4}+\ldots+x^4+x^2+1=0,$$

a recurring equation, having $n-2$ imaginary roots.

For example, the equation $x^6=1$, or $x^6-1=0$ divided by x^2-1 gives $x^4+x^2+1=0$; whence,

$$x=\pm\sqrt{\left\{\frac{-1\pm\sqrt{-3}}{2}\right\}}.$$

This gives for the six roots of 1

$$+1, \qquad -1,$$

$$+\sqrt{\frac{-1+\sqrt{-3}}{2}}, \quad -\sqrt{\frac{-1+\sqrt{-3}}{2}},$$

$$+\sqrt{\frac{-1-\sqrt{-3}}{2}}, \quad -\sqrt{\frac{-1-\sqrt{-3}}{2}}.$$

III.—If n be odd, the equation $x^n-1=0$ has only one real root, viz.: $+1$; for $+1$ is the only real number of which the odd powers are $+1$.

Dividing $x^n-1=0$ by $x-1$, we have

$$x^{n-1}+x^{n-2}+x^{n-3}+\ldots\ldots+x^2+x+1=0,$$

a recurring equation, having $n-1$ imaginary roots.

For example, the equation $x^3=1$, or $x^3-1=0$, divided by $x-1$, gives $x^2+x+1=0$; whence, $x=\frac{-1\pm\sqrt{-3}}{2}$.

Hence, the three third roots of 1 are

$$1, \qquad \frac{-1+\sqrt{-3}}{2}, \quad \frac{-1-\sqrt{-3}}{2}.$$

IV.—If n be even, the equation $x^n+1=0$, or $x^n=-1$, has no real root, since $\sqrt[n]{-1}$ is then impossible. Hence, all the roots of this equation are imaginary.

For example, the four roots of the recurring equation $x^4+1=0$ (Art. 442), are

$$\frac{-1+\sqrt{-1}}{\sqrt{2}},\quad \frac{-1-\sqrt{-1}}{\sqrt{2}},\quad \frac{1+\sqrt{-1}}{\sqrt{2}},\quad \frac{1-\sqrt{-1}}{\sqrt{2}}.$$

V.—If n be odd, the equation $x^n+1=0$, or $x^n=-1$, has one real root, viz.: -1, and no more, because this is the only real number of which an odd power is -1.

For example, $x^3+1=0$, divided by $x+1$, gives $x^2-x+1=0$; whence, $x=\dfrac{1\pm\sqrt{-3}}{2}$.

Therefore, the three third roots of -1, are -1, $\dfrac{1+\sqrt{-3}}{2}$, and $\dfrac{1-\sqrt{-3}}{2}$.

Binomial equations have other properties, but some of them can not be discussed without a knowledge of Analytical Trigonometry.

1. Find the four fourth roots of unity.

Ans. $+1$, -1, $+\sqrt{-1}$, $-\sqrt{-1}$.

2. Find the five fifth roots of unity.

Ans. 1,

$\frac{1}{4}\{\sqrt{5}-1+\sqrt{(-10-2\sqrt{5})}\}$,

$\frac{1}{4}\{\sqrt{5}-1-\sqrt{(-10-2\sqrt{5})}\}$,

$-\frac{1}{4}\{\sqrt{5}+1+\sqrt{(-10+2\sqrt{5})}\}$,

$-\frac{1}{4}\{\sqrt{5}+1-\sqrt{(-10+2\sqrt{5})}\}$.

THE END.